The Rise of the Machines: Automation, Horizontal Innovation and Income Inequality

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June 2016 (first draft: September 2013)

Abstract

We construct an endogenous growth model with automation (the introduction of machines which replace low-skill labor) and horizontal innovation. The economy follows three phases. First, low-skill wages are low, which induces little automation, and income inequality and labor's share of GDP are constant. Second, as low-skill wages increase, automation increases which reduces the labor share, increases the skill premium and may decrease future low-skill wages. Finally, the economy moves toward a steady state, where low-skill wages grow but at a lower rate than high-skill wages. Surprisingly, a more productive automation technology increases low-skill wages in the long-run.

JEL: O41, O31, O33, E23, E25.

KEYWORDS: Endogenous growth, automation, horizontal innovation, directed technical change, income inequality.

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1 Introduction

How does the automation of the production process drive economic growth and affect the distribution of income? Conversely, how do wages shape technological progress? The last 40 years in particular have seen dramatic changes in the income distribution with the skill premium rising throughout, low-skill wages stagnating and more recently a phase of wage polarization. These changes are often attributed to skill-biased technical change: by allowing for the use of machines in some tasks, automation increases economic output, but it also reduces the demand for certain types of labor, particularly low-skill labor. Autor, Levy and Murnane (2003) among others provide evidence in support of this mechanism. As the range of tasks that machines can perform has expanded considerably, the general public is increasingly worried about the negative consequences of technological progress. Yet, economists often argue that technological development also creates new products and tasks, which boost the demand for labor; and certainly many of today's jobs did not exist just a few decades ago.[1] Surprisingly, the economics literature lacks a dynamic framework to analyze the interaction between automation and the creation of new products. This paper provides the first model that can do so.

Of course, a large literature exists which relates exogenous technical change to the income distribution (e.g. Goldin and Katz, 2008, and Krusell, Ohanian, Ríos-Rull and Violante, 2002). Previous attempts at endogenizing the direction of technical change rely on factor-augmentation and exogenous shocks to the skill supply (Acemoglu, 1998). The novelty of our approach is that we present an endogenous growth version of a task framework in the vein of Autor, Levy and Murnane (2003) and Acemoglu and Autor (2011), in which the direction of innovation evolves endogenously.

The main lesson from our framework is the following. If tasks performed by a scarce factor (say labor) can potentially be automated but it is not presently profitable to do so, then, in a growing economy, the return to this factor will eventually increase sufficiently to make it profitable. Once automation has been triggered, the economy endogenously transitions from one aggregate production function to another and during this transition factor returns might drop. We characterize when this might happen and show that this decrease must be temporary. Although, we focus on a general equilibrium model with low-skill labor, the insights extend to subsectors of the economy and other scarce factors.

We consider an expanding variety growth model with low-skill and high-skill workers.

[1] For instance, the introduction of the telephone led to the creation of new jobs. In 1970 there were 421 000 switchboard operators in the United States. This occupation has largely been automated today.

Horizontal innovation, modeled as in Romer (1990), increases the demand for both low- and high-skill workers. Automation allows for the replacement of low-skill workers with machines in production. It takes the form of a secondary innovation in existing product lines, similar to the secondary innovations in Aghion and Howitt (1996) (though their focus is on the interplay between applied and fundamental research, and not on automation). Within a firm, automation increases the demand for high-skill workers but reduces the demand for low-skill workers. "Non-automated" products only use low-skill and high-skill labor. Once invented, a specific machine is produced with the same technology as the consumption good.

We first take the level of technology as exogenous and show that an increase in the number of varieties will increase all wages. An increase in automation, however, will have the dual effect of increasing the overall productivity of the economy and allowing substitution away from low-skill workers, resulting in an ambiguous net effect on low-skill wages. Nevertheless, we show that for very general processes of horizontal and automation innovation, the asymptotic growth rate of low-skill wages must be positive, albeit strictly lower than that of high-skill wages.

Having studied the impact of technological change on wages, we require innovation to be the process of deliberate investment and show the key role played by low-skill wages. The cost advantage of an automated over a non-automated firm increases with the real wage of low-skill workers, so that the incentive to automate is not constant over time. As a consequence, the economy does not support a balanced growth path. Instead, an economy with an initially low level of technology first goes through a phase where growth is mostly generated by horizontal innovation and the skill premium and the labor share are constant. Only when low-skill wages are sufficiently high will firms invest in automation. During this second phase—where our model differs most from the existing literature—the share of automated products increases, the skill premium rises and, depending on parameters, the real low-skill wage may temporarily decrease. The total labor share decreases progressively, in line with recent evidence (Karabarbounis and Neiman, 2013). Finally, the economy moves towards its asymptotic steady state. The share of automated products stabilizes as the entry of new, non-automated products compensates for the automation of existing ones. The total labor share stabilizes. Eventually, the economy will have endogenously shifted from a Cobb-Douglas aggregate production function to a nested CES one.

A simpler capital deepening model without automation innovation, but where low-skill labor and capital are always substitutes would also feature a secular rise in the skill premium and a drop in the labor share. Yet our analysis shows that there are several features which distinguish our model from previous work. First, contrary to a capital deepening model, we can generate stagnating or even declining real low-skill wages along the equilibrium path. Second, we can analyze the *interaction* between automation and another form of technological progress. For instance, we obtain the surprising result that a more productive automation technology increases the long-run growth rate of low-skill wages because it encourages horizontal innovation—although it may lead to lower low-skill wages for some time. This simple comparative statics result speaks to potential differences in policy implications between this setup and one without endogenous technical change. Third, by analyzing innovation patterns, our set-up makes some of the misconceptions that arise from restricting attention to factor-augmenting models apparent: for instance, intense automation can be consistent with a decline in labor productivity growth and the impact of automation on low-skill wages and the skill-premium need not be the strongest when expenses on automation are the largest; a response to two points of critique of the skill-biased technological change hypothesis put forward by Card and DiNardo (2002). In addition, while the elasticity of substitution between factors is of central importance for the labor share in factor-augmenting models (Piketty, 2014 and Karabarbounis and Neiman, 2014), our model highlights the role played by the share parameters.

Then, we extend the baseline model to include a supply response in the skill distribution, and calibrate it to match the evolution of the skill premium, the skill ratio, the labor share and productivity growth since the 1960s. As is common in the literature, for this exercise (and only this exercise) we identify skill groups with education groups, such that high-skill workers correspond to college-educated workers. This exercise demonstrates that our model is able to replicate the trends in the data quantitatively, even though we do not feed in any input time paths from the data as is usually done.

Finally, recent empirical work has increasingly found that workers in the middle of the income distribution are most adversely affected by technological progress. To address this, we extend the baseline model to include middle-skill workers as a separate skill-group. Products either rely on low-skill or middle-skill workers and the two skill-groups are symmetric except that automating to replace middle-skill workers is more costly (or alternatively machines are less productive in middle-skill firms). This implies that the automation of low-skill workers' tasks happens first, with a delayed automation process for the tasks of middle-skill workers. We show that this difference can reproduce important trends in the United States income distribution: In a first period, there is a uniform dispersion of the income distribution, as low-skill workers' products are rapidly automated but middle-skill ones are not; while in the second period there is wage polarization: low-skill workers' share of automated products is stabilized, and middle-skill products are more rapidly automated.

Our modeling of automation as a skill-biased innovation is motivated by a large empirical literature. For instance, Autor, Katz and Krueger (1998) and Autor, Levy and Murnane (2003) use cross-sectional data to demonstrate that computerization is associated with relative shifts in demand favoring college-educated workers, Bartel, Ichniowski and Shaw (2007) present similar evidence at the firm level. Similarly, Graetz and Michaels (2015) show that the introduction of industrial robots leads to a reduction in the demand for low- (and middle-) skill workers. The idea that high wages could incentivize technological progress in the form of automation dates back to Habakkuk (1962). The empirical literature on this relationship is more modest, but Lewis (2001) finds that low-skill immigration slows down the adoption of automation technology and Hornbeck and Naidu (2014) find that the emigration of black workers from the American South favored the adoption of modern agricultural production techniques.

There is a small theoretical literature on labor-replacing technology. In Zeira (1998), exogenous increases in TFP raise wages and encourage the adoption of a capital-intensive technology analogous to automation in this paper. Acemoglu (2010) shows that labor scarcity induces innovation (the Habbakuk hypothesis), if and only if innovation is labor-saving, that is, if it reduces the marginal product of labor. Neither paper analyzes labor-replacing innovation in a fully dynamic model nor focuses on income inequality, as we do. Peretto and Seater (2013) build a dynamic model of factor-eliminating technical change where firms learn how to replace labor with capital. Since wages are constant the incentive to automate does not change over time. In addition, they do not focus on income inequality. Benzell, Kotlikoff, LaGarda and Sachs (2015), following Sachs and Kotlikoff (2012), build an overlapping generation model where a code-capital stock can substitute for labor. A technological shock which favors the accumulation of code-capital can lead to lower long-run GDP by reducing wages and thereby investment in physical capital. In both papers (and contrary to our model), the technological shock is completely exogenous. Finally, in work subsequent to our paper, Acemoglu and Restrepo (2015) also develop a growth model where technical changes involves automation and the creation of new tasks. We discuss their model in details in section 3.8.

A large literature has used skill-biased technical change (SBTC) as a possible explanation for the increase in the skill premium in developed countries since the 1970's, despite a large increase in the relative supply of skilled workers (see Hornstein, Krusell and Violante, 2005, for a more complete literature review). One can categorize theoretical papers into one of three strands. The first strand emphasizes the hypothesis of Nelson and Phelps (1966) that more skilled workers are better able to adapt to technological change (see Lloyd-Ellis, 1999; Caselli, 1999; Galor and Moav, 2000, and Aghion, Howitt and Violante, 2002). However, such theories mostly explain transitory increases in inequality whereas inequality has been increasing for decades. Our model, on the contrary, introduces a mechanism that creates permanent and widening inequality.

A second strand sees the complementarity between capital and skill as the source for the increase in the skill premium. Krusell, Ohanian, Ríos-Rull and Violante (2000) develop a framework where capital equipment and high-skill labor are complements. By adding the observed increase in the stock of capital equipment, they can account for most of the variation in the skill premium. Our model shares features with their framework: machines play an analogous role to capital equipment in their model, since they are more complementary with high-skill labor than with low-skill labor. The focus of our paper is different though since we seek to explain *why* innovation has been directed towards automation, and analyze the interactions between automation and horizontal innovation.

Finally, a third branch of the literature, building on Katz and Murphy (1992), considers technology to be either high-skill or low-skill labor augmenting and infers the bias of technology from changes in the relative supply and the skill-premium. Goldin and Katz (2008) employ this framework to conclude that technical change has been skill-biased throughout the 20^{th} century in the United States (Katz and Margo, 2014, argue that the relative demand for white-collar workers has been increasing since 1820). This work, however, does not attempt to endogenize the skill bias of technical change. This is done in the (more theoretical) directed technical change literature (most notably Acemoglu, 1998, 2002 and 2007). Such models, which also use factor-augmenting technical change, deliver important insights about inequality and technical change, but they have no role for labor-replacing technology (a point emphasized in Acemoglu and Autor, 2011). In addition, even though income inequality varies, wages cannot decrease in absolute terms, and their asymptotic growth rates must be the same. The present model is also a directed technical change framework as economic incentives determine the form that technical change takes, but it deviates from the assumption of factor-augmenting technologies and explicitly allows for labor-replacing automation, generating the possibility for (temporary) absolute losses for low-skill workers, and permanently increasing income inequality.

More recently, Autor, Katz, and Kearney (2006, 2008) and Autor and Dorn (2013), amongst others, show that whereas income inequality has continued to increase above the median, there has been a reversal below the median. They argue that the routine tasks performed by many middle-skill workers—storing, processing and retrieving information—are more easily done by computers than those performed by low-skill workers, now predominantly working in service occupations. This "wage polarization" has been accompanied by a "job polarization" as employment has followed the same pattern of decreasing employment in middle-skill occupations.[2] Our explanation is related but distinct: it is because low-skill tasks have already been heavily automated that automation is now more prominent in middle-skill tasks. Hence, we provide a unified explanation for the relative decline of middle-skill wages since the mid-1980s and the relative decline of low-skill wages in the period before.

The paper proceeds as follows: Section 2 describes the baseline model for exogenous technological change, it shows the consequences of technological change on wages and derives the asymptotic behavior. Section 3 endogenizes the path of technological change and describes the evolution of the economy through three phases. Section 4 calibrates an extended version of the model (with an endogenous labor supply response in the skill distribution) to the US economy since the 1960s. Section 5 extends the model to analyze wage polarization. Section 6 concludes.

2 A Baseline Model with Exogenous Innovation

2.1 Preferences and production

We consider a continuous time infinite-horizon economy populated by H high-skill and L low-skill workers. Both types of workers supply labor inelastically and have identical

[2] This phenomenon has also been observed and associated with the automation of routine tasks in Europe (Spitz-Oener, 2006, and Goos, Manning and Salomons, 2009). A related literature analyzes this non-monotonic pattern in inequality changes through the lens of assignment models where workers of different skill levels are matched to tasks of different skill productivities (e.g. Costinot and Vogel, 2010; Burstein, Morales and Vogel, 2014 and Feng and Graetz, 2015).

preferences over a single final good of:

$$U_{k,t} = \int_t^\infty e^{-\rho(\tau - t)} \frac{C_{k,\tau}^{1-\theta}}{1-\theta} d\tau,$$

where ρ is the discount rate, $\theta \geq 1$ is the inverse elasticity of intertemporal substitution and $C_{k,t}$ is consumption of the final good at time t by group $k \in \{H, L\}$. H and L are kept constant in our baseline model, but we consider the case where workers choose occupations based on relative wages and heterogeneous skill-endowments in Section 4.1.

The final good is produced by a competitive industry combining an endogenous set of intermediate inputs, $i \in [0, N_t]$ using a CES aggregator:

$$Y_t = \left(\int_0^{N_t} y_t(i)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}},$$

where $\sigma > 1$ is the elasticity of substitution between these inputs and $y_t(i)$ is the use of intermediate input i at time t. As in Romer (1990), an increase in N_t represents a source of technological progress. Throughout the paper, we use interchangeably the terms "intermediate input" and "product".

We normalize the price of Y_t to 1 at all points in time and drop time subscripts when there is no ambiguity. The demand for each intermediate input i is:

$$y(i) = p(i)^{-\sigma} Y, \tag{1}$$

where $p(i)$ is the price of intermediate input i and the normalization implies that the ideal price index, $[\int_0^N p(i)^{1-\sigma} di]^{1/(1-\sigma)}$ equals 1.

Each intermediate input is produced by a monopolist who owns the perpetual rights of production. She can produce the intermediate input by combining low-skill labor, $l(i)$, high-skill labor, $h(i)$, and, possibly, type-i machines, $x(i)$, using the production function:

$$y(i) = \left[l(i)^{\frac{\epsilon-1}{\epsilon}} + \alpha(i) \left(\tilde{\varphi} x(i) \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon\beta}{\epsilon-1}} h(i)^{1-\beta}, \tag{2}$$

where $\alpha(i) \in \{0, 1\}$ is an indicator function for whether or not the monopolist has access to an automation technology which allows for the use of machines. If the product is not automated ($\alpha(i) = 0$), production takes place using a Cobb-Douglas production function with only low-skill and high-skill labor with a low-skill factor share of β. If the product is automated ($\alpha(i) = 1$) machines can be used in the production process. We allow for perfect substitutability, in which case $\epsilon = \infty$ and the production function is $y(i) = [l(i) + \alpha(i)\tilde{\varphi}x(i)]^{\beta} h(i)^{1-\beta}$. The parameter $\tilde{\varphi}$ is the relative productivity advantage of machines over low-skill workers and G denotes the share of automated products.[3]

Since each input is produced by a single firm, from now on we identify each input with its firm and we refer to a firm which produces an automated product as an automated firm. We refer to the specific labor inputs provided by high-skill and low-skill workers in the production of different inputs as "different tasks" performed by these workers, so that each product comes with its own tasks. It is because $\alpha(i)$ is not fixed, but can change over time, that our model captures the notion that machines can replace low-skill labor in new tasks. A model in which $\alpha(i)$ were fixed for each product would only allow for machines to be used more intensively in production, but always for the same tasks.

Throughout the paper we will refer to x as "machines", though our interpretation also includes any form of computer inputs, algorithms, the services of cloud-providers, etc. For simplicity, we consider that machines depreciate immediately, but Appendix 7.4 relaxes this assumption. Once invented, machines of type i are produced competitively one for one with the final good, so that the price of an *existing* machine for an automated firm is always equal to 1—hence technological progress in machine production follows that in the rest of the economy. Though a natural starting point, this is an important assumption and Appendix 7.3 presents a version of the model which relaxes it. Nevertheless, we stress that our model can capture the notion of a decline in the real cost of equipment: indeed automation for firm i can equivalently be interpreted as a decline of the price of machines i from infinity to 1.

2.2 Equilibrium wages

In this section we take the technological levels N (the number of products) and G (the share of automated products) as well as the employment of high-skill workers in production, H^P as given (we will let $H^P \leq H$ to accommodate later sections where high-skill labor is used to innovate). We now derive how wages are determined in equilibrium.

First, note that all automated firms are symmetric and therefore behave in the same

[3]Following existing evidence (Autor, Katz and Krueger, 1998, Autor, Levy and Murnane, 2003, or Bartel, Ichniowski and Shaw, 2007), we assume that the high-skill tasks cannot be automated. Yet, we discuss the automation of these tasks later in the paper.

way. Similarly all non-automated firms are symmetric. This gives aggregate output of:

$$Y = N^{\frac{1}{\sigma-1}} \times$$

$$\left((1-G)^{\frac{1}{\sigma}} \Big(\underbrace{\left(L^{NA}\right)^{\beta} \left(H^{P,NA}\right)^{1-\beta}}_{T_1} \Big)^{\frac{\sigma-1}{\sigma}} + G^{\frac{1}{\sigma}} \Big(\underbrace{\left[\left(L^{A}\right)^{\frac{\epsilon-1}{\epsilon}} + (\tilde{\varphi} X)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon\beta}{\epsilon-1}} \left(H^{P,A}\right)^{1-\beta}}_{T_2} \Big)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{3}$$

where L^A (respectively L^{NA}) is the total mass of low-skill workers in automated (respectively non-automated) firms, $H^{P,A}$ (respectively $H^{P,NA}$) is the total mass of high-skill workers hired in production in automated (respectively non-automated) firms and $X = \int_0^N x(i)di$ is total use of machines. The aggregate production function takes the form of a nested CES between two sub-production functions. The first term T_1 captures the classic case where production takes place with constant shares between factors (low-skill and high-skill labor), while the second term T_2 represents the factors used within automated products and features the substitutability between low-skill labor and machines. G is the share parameter of the "automated" products nest and therefore an increase in G is T_2-biased (as $\sigma > 1$). $N^{\frac{1}{\sigma-1}}$ is a TFP parameter. Besides the functional form the aggregate production function (3) differs from the often assumed aggregate CES production function in two ways: first, it is derived from the cost functions of individual firms and the technological change that we consider (new products and machines undertaking more tasks) is more concrete than the usual factor-augmenting technical change. Second, once we endogenize G we will be able to capture effects that the usual focus on an exogenous aggregate production function cannot.

The unit cost of intermediate input i is given by:

$$c(w, v, \alpha(i)) = \beta^{-\beta}(1-\beta)^{-(1-\beta)} \left(w^{1-\epsilon} + \varphi\alpha(i)\right)^{\frac{\beta}{1-\epsilon}} v^{1-\beta}, \tag{4}$$

where $\varphi \equiv \tilde{\varphi}^{\epsilon}$, w denotes low-skill wages and v high-skill wages. When $\epsilon < \infty$, $c(\cdot)$ is strictly increasing in both w and v and $c(w, v, 1) < c(w, v, 0)$ for all $w, v > 0$ (automation reduces costs). The monopolist charges a constant markup over costs such that the price is $p(i) = \sigma/(\sigma-1) \cdot c(w, v, \alpha(i))$.

Using Shepard's lemma and equations (1) and (4) delivers the demand for low-skill labor of a single firm.

$$l(w,v,\alpha(i)) = \beta \frac{w^{-\epsilon}}{w^{1-\epsilon}+\varphi\alpha(i)} \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} c(w,v,\alpha(i))^{1-\sigma} Y, \tag{5}$$

which is decreasing in w and v. The effect of automation on demand for low-skill labor in a given firm is generally ambiguous. This is due to the combination of a negative *substitution* effect (the ability of the firm to substitute machines for low-skill workers) and a positive *scale* effect (the ability of the firm to employ machines decreases overall costs, lowers prices and increases production). Since our focus is on labor-substituting innovation, we impose the condition $\epsilon > 1+\beta(\sigma-1)$ *throughout* the paper which is both necessary and sufficient for the substitution effect to dominate and ensure $l(w,v,1) < l(w,v,0)$ for all $w, v > 0$.

Let $x(w,v)$ denote the use of machines by an automated firm. The relative use of machines and low-skill labor for such a firm is then:

$$x(w,v)/l(w,v,1) = \varphi w^{\epsilon}, \tag{6}$$

which is increasing in w as the real wage is also the price of low-skill labor relative to machines.

The iso-elastic demand (1), coupled with constant mark-up $\sigma/(\sigma-1)$, implies that revenues are given by $R(w,v,\alpha(i)) = ((\sigma-1)/\sigma)^{\sigma-1} c(w,v,\alpha(i))^{1-\sigma} Y$ and that a share $1/\sigma$ of revenues accrues to the monopolists as profits: $\pi(w,v,\alpha(i)) = R(w,v,\alpha(i))/\sigma$. Aggregate profits are then a constant share $1/\sigma$ of output Y, since output is equal to the aggregate revenues of intermediate inputs firms. We define $\mu \equiv \beta(\sigma-1)/(\epsilon-1) < 1$ (by our assumption on ϵ). Using (4), the relative revenues (and profits) of non-automated and automated firms are given by:

$$\frac{R(w,v,0)}{R(w,v,1)} = \frac{\pi(w,v,0)}{\pi(w,v,1)} = \left(1+\varphi w^{\epsilon-1}\right)^{-\mu}, \tag{7}$$

which is a decreasing function of w. Since non-automated firms rely more heavily on low-skill labor, their relative market share drops with higher low-skill wages.

The share of revenues in a firm accruing to high-skill labor in production is the same whether a firm is automated or not and given by $\nu_h = (1-\beta)(\sigma-1)/\sigma$. Using labor market clearing for high-skill workers in production ($\int_0^N h(i)di = H^P$) and aggregating over those workers, we get that

$$vH^P = (1-\beta)\frac{\sigma-1}{\sigma}N\left[GR(w,v,1)+(1-G)R(w,v,0)\right] = (1-\beta)\frac{\sigma-1}{\sigma}Y. \tag{8}$$

Using factor demand functions, the share of revenues accruing to low-skill labor is given by $\nu_l(w,v,\alpha(i)) = \frac{\sigma-1}{\sigma}\beta\left(1+\varphi w^{\epsilon-1}\alpha(i)\right)^{-1}$, and is lower for automated than non-automated firms. Similarly using low-skill labor market clearing ($\int_0^N l(i)di = L$), we obtain the aggregate revenues of low-skill workers as:

$$wL = N\left[GR(w,v,1)\nu_l(w,v,1)+(1-G)R(w,v,0)\nu_l(w,v,0)\right]. \tag{9}$$

Taking the ratio of (8) over (9) and using (7) gives the following lemma.

Lemma 1. *For $\epsilon < \infty$, the high-skill wage premium is given by*[4]

$$\frac{v}{w} = \frac{1-\beta}{\beta}\frac{L}{H^P}\frac{G+(1-G)(1+\varphi w^{\epsilon-1})^{-\mu}}{G\left(1+\varphi w^{\epsilon-1}\right)^{-1}+(1-G)(1+\varphi w^{\epsilon-1})^{-\mu}}. \tag{10}$$

For given L/H^P and $G>0$, the skill premium is increasing in the absolute level of low-skill wages, which means that if G is bounded above 0, low-skill wages cannot grow at the same rate as high-skill wages in the long-run. This is the case because higher low-skill wages both induce more substitution towards machines in automated firms (as reflected by the term $(1+\varphi w^{\epsilon-1})^{-1}$ in equation (10)) and improve the cost-advantage and therefore the market share of automated firms (term $(1+\varphi w^{\epsilon-1})^{-\mu}$), which rely relatively less on low-skill workers.

With constant mark-ups, the cost equation (4) and the price normalization give:

$$\frac{\sigma}{\sigma-1}\frac{N^{\frac{1}{1-\sigma}}}{\beta^\beta\left(1-\beta\right)^{1-\beta}}\left(G\left(\varphi+w^{1-\epsilon}\right)^{\mu}+(1-G)w^{\beta(1-\sigma)}\right)^{\frac{1}{1-\sigma}}v^{1-\beta}=1. \tag{11}$$

This *productivity* condition shows the positive relationship between real wages and the level of technology given by N, the number of intermediate inputs, and G the share of automated firms. Together (10) and (11) determine real wages uniquely as a function of technology N, G and the mass of high-skill workers engaged in production H^P.

Though the production function implies that, at the firm level, the elasticity of substitution between high-skill labor and machines is equal to that between high-skill

[4]When machines and low-skill workers are perfect substitutes, $\epsilon=\infty$, the skill premium is given by $\frac{v}{w}=\frac{1-\beta}{\beta}\frac{L}{H^P}$ if $w<\widetilde{\varphi}^{-1}$ such that no firm uses machines, and $\frac{v}{w}=\frac{1-\beta}{\beta}\frac{L}{H^P}\frac{G+(1-G)(\widetilde{\varphi}w)^{-1}}{(1-G)(\widetilde{\varphi}w)^{-1}}$ if $w>\widetilde{\varphi}^{-1}$.

and low-skill labor, this does not imply that the same holds at the aggregate level. Therefore our paper is not in contradiction with Krusell et al. (2000), who argue that the aggregate elasticity of substitution between high-skill and low-skill labor is greater than the one between high-skill labor and machines.[5]

Given the amount of resources devoted to production (L, H^P), the static equilibrium is closed by the final good market clearing condition:

$$Y = C + X \tag{12}$$

where $C = C_L + C_H$ is total consumption.

2.3 Technological change and wages

The consequences of technological changes on the level of wages are most easily seen with the help of Figure 1 which plots the skill-premium (10) and productivity (11) conditions in (w, v) space.[6]

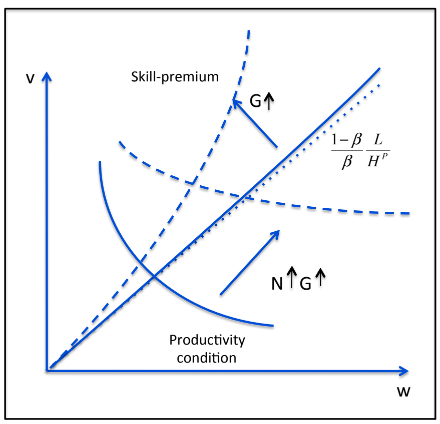

Figure 1: Evolution of high-skill (v) and low-skill (w) wages following technological changes. An increase in N pushes out the productivity condition increasing both wages. An increase in G pushes out the productivity condition and pivots the skill-premium counter-clockwise, which increases v but has an ambiguous effect on w.

An increase in the number of products, N, pushes out the productivity condition which results in an increase in both low-skill and high-skill wages. When $G = 0$, both

[5]This illustrates the difference between assuming elasticities of substitution of the aggregate production function and deriving them from elasticities of substitution at the firm level. In fact, in our model, the aggregate elasticities of substitution are not constant (see Appendix 8.3.5).

[6]When $\epsilon = \infty$ and $G = 1$, the productivity condition has an horizontal arm and the skill-premium condition a vertical one.

types of wages grow at the same rate as the skill premium condition is a straight line with slope $(1-\beta)L/(\beta H^P)$. For $G>0$, the skill premium curve is not linear and high-skill wages grow proportionally more than low-skill wages (following lemma 1).

An increase in the share of automated products G has a positive effect on high-skill wages and the skill premium with an ambiguous effect on low-skill wages. Indeed, higher automation increases the productive capability of the economy and pushes out the productivity condition (an *aggregate productivity* effect), but it also allows the economy to more easily substitute away from low-skill labor which pivots the skill-premium condition counter-clockwise (an *aggregate substitution*). The former effect increases low-skill wages and the latter decreases low-skill wages. Therefore automation is low-skill labor saving (in the vocabulary of Acemoglu, 2010) if and only if the aggregate substitution effect dominates the aggregate productivity effect. The following proposition (derived in Appendix 8.1) gives the full comparative statics.

Proposition 1. *Consider the equilibrium (w,v) determined by equations (10) and (11). It holds that*

A) An increase in the number of products N (keeping G and H^P constant) leads to an increase in both high-skill (v) and low-skill wages (w). Provided that $G>0$, an increase in N also increases the skill premium v/w.

B) An increase in the share of automated products G (keeping N and H^P constant) increases the high-skill wages v and the skill premium v/w. Furthermore,

- i) if $1/(1-\beta)\leq\sigma-1$, low-skill wages w are decreasing in G;

- ii) if $\sigma-1<1/(1-\beta)<(\epsilon-1)/\beta$, w is decreasing in G for N sufficiently low and inversely u-shaped in G otherwise, with $w|_{G=1}<w|_{G=0}$;

- (iii) if $1/(1-\beta)=(\epsilon-1)/\beta$, w is inversely u-shaped in G with $w|_{G=1}=w|_{G=0}$;

- (iv) if $(\epsilon-1)/\beta<1/(1-\beta)$, w is increasing in G for N sufficiently low and inversely u-shaped otherwise (weakly if $\epsilon=\infty$), with $w|_{G=1}>w|_{G=0}$.

The proposition states that $\beta/(1-\beta)<\epsilon-1$ — such that the elasticity of substitution between machines and low-skill workers is sufficiently large — is a necessary and sufficient condition to ensure that low-skill wages are lower in a world where all products are automated than in a world where none are. A low cost share of low-skill workers/machines, β, will make this more likely as automation then provides less cost savings and a lower aggregate productivity effect. For $\sigma-1<1/(1-\beta)$ and N (and therefore low-skill wages) sufficiently large, w is inversely u-shaped in G. This is because the automation of the first products has a relatively large productivity effect on the economy, but a relatively small aggregate substitution effect since most firms are still non-automated, whereas the converse is true once G is sufficiently high.

In section 3, when we specify the innovation process, we will show that as the number of products N increases, the economy endogenously experiences a change in the share of automated products: from a low level, close to 0, to a higher level. As a result, growth will progressively become unbalanced with a rising skill premium, and for some parameter values, low-skill wages may temporarily decline.

2.4 Asymptotics for general technological processes

We study the asymptotic behavior of the model for given paths of technologies and mass of high-skill workers in production. For any variable a_t (such as N_t), we let $g_t^a \equiv \dot{a}_t/a_t$ denote its growth rate and $g_\infty^a = lim_{t\to\infty} g_t^a$ if it exists. In Appendix 8.2.1 we derive

Proposition 2. *Consider three processes $[N_t]_{t=0}^\infty$, $[G_t]_{t=0}^\infty$ and $[H_t^P]_{t=0}^\infty$ where $(N_t, G_t, H_t^P) \in (0,\infty) \times [0,1] \times (0,H]$ for all t. Assume that G_t, g_t^N and H_t^P all admit strictly positive limits G_∞, g_∞^N and H_∞^P. Then, the growth rates of high-skill wages and output admit limits with:*

$$g_\infty^v = g_\infty^Y = g_\infty^N / \left((1-\beta)(\sigma-1)\right). \tag{13}$$

Part A) If $0 < G_\infty < 1$ then the asymptotic growth rate of w_t is given by

$$g_\infty^w = g_\infty^Y / \left(1 + \beta(\sigma-1)\right). \tag{14}$$

Part B). If $G_\infty = 1$ and G_t converges sufficiently fast (more specifically if $lim_{t\to\infty} \left(1-G_t\right) N_t^{\psi(1-\mu)\frac{\epsilon-1}{\epsilon}}$ exists and is finite) then :

- If $\epsilon < \infty$ the asymptotic growth rate of w_t is positive at :

$$g_\infty^w = g_\infty^Y / \epsilon, \tag{15}$$

where $1 + \beta(\sigma-1) < \epsilon$ by assumption.[7]

- If low-skill workers and machines are perfect substitutes then $lim_{t\to\infty} w_t$ is finite and weakly greater than $\tilde{\varphi}^{-1}$ (equal to $\tilde{\varphi}^{-1}$ when $lim_{t\to\infty} \left(1-G_t\right) N_t^\psi = 0$).

This proposition first relates the growth rate of output and high-skill wages to the growth rate of the number of products. Without automation Y_t would be proportional

[7] If $lim_{t\to\infty} \left(1-G_t\right) N_t^{\psi(1-\mu)\frac{\epsilon-1}{\epsilon}} = \infty$ then $g_\infty^Y/\epsilon \le g_\infty^w \le g_\infty^Y / \left(1+\beta(\sigma-1)\right)$.

to $N_t^{1/(\sigma-1)}$, as in a standard expanding-variety model: the higher the degree of substitutability between inputs the lower the gain in productivity from an increase in N_t. Here, because automation allows the use of machines as an additional input, there is an acceleration effect as the higher productivity also increases the supply of machines (as long as $G_\infty > 0$). Asymptotically, this effect is increasing in the factor share of low-skill workers/machines, β.[8]

Second, this proposition shows that, with positive growth in N_t, mild assumptions are sufficient to guarantee an asymptotic positive growth rate for w_t: in line with Proposition 1 the only case in which w_t would not be increasing occurs when G_t converges to 1 sufficiently fast and low-skill workers and machines are perfect substitutes. To gain further intuition, first consider the case in which $G_\infty < 1$. Since automated and non-automated products are imperfect substitutes, then so are machines and low-skill workers at the aggregate level. As the aggregate production function is a nested CES with asymptotically constant weights (see equation (3)), a growing stock of machines and a fixed supply of low-skill labor, implies that the relative price of a worker (w_t) to a machine (p_t^x) must grow at a positive rate. Since machines are produced with the same technology as the consumption good, $p_t^x = p_t^C$, where p_t^C is the price of the consumption good (1 with our normalization), and the real wage $w_t = w_t/p_t^C = (w_t/p_t^x)(p_t^x/p_t^C)$ must also grow at a positive rate.[9]

With growing wages, the relative market share of automated firms and their reliance on machines increase, which ensure that low-skill wages grow at a lower rate than the economy (with $G_\infty > 0$ the assumptions of Uzawa's theorem are not satisfied since horizontal innovation is not low-skill labor augmenting). Under our assumption that automation is labor-saving at the firm level ($\epsilon < 1 + \beta(\sigma - 1)$), the demand for low-skill labor increasingly comes from the non-automated firms. As a result, the ratio between high-skill and low-skill wages growth rates increases with a higher importance of low-skill workers (a higher β) or a higher substitutability between automated and non-automated products (a higher σ) since both imply a faster loss of competitiveness of the non-automated firms. On the other hand, it is independent of the elasticity of substitution between machines and low-skill workers, ϵ.

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[8] Equation (13) also makes it clear why we must impose $\beta < 1$. If $\beta = 1$ the economy reaches a "world of plenty" in finite time, as infinite production is possible once the number of products N is sufficiently large relative to the productivity parameter $\widetilde{\varphi}$. In reality one may think that other factors such as natural resources or land would then become the scarce factor.

[9] A generalized version of Proposition 2 is presented in Appendix 7.3 which allows for asymptotic (negative) growth in p_t^x/p_t^C and thereby potentially decreasing real wages for low-skill workers.

Now, consider the case of $G_\infty = 1$ and $\epsilon < \infty$ (and let convergence satisfy the condition in Part B of Proposition 2). Then an analogous argument demonstrates that low-skill wages must increase asymptotically, though the growth rate relative to that of the economy must be lower than when $G_\infty < 1$ as all firms are automated and automated firms more readily substitute workers for machines than the economy substitutes from non-automated to automated products. The more easily they substitute (the higher is ϵ) the lower the growth rate of low-skill workers wages. Only in the special case in which machines and low-skill workers are perfect substitutes in the production by automated firms and the share of automated firms is asymptotically 1 will there be economy-wide perfect substitution between low-skill workers and machines. In this case, w_t cannot grow asymptotically, but will still be bounded below by $\tilde{\varphi}^{-1}$, since a lower wage would imply that no firm would use machines.[10]

In general, the processes of N_t, G_t and H_t^P will depend on the rate at which new products are introduced, the extent to which they are initially automated, and the rate at which non-automated firms are automated. The following lemma derives condition under which $G_\infty < 1$, so that Part A of Proposition 2 applies.

Lemma 2. *Consider processes $[N_t]_{t=0}^{\infty}$, $[G_t]_{t=0}^{\infty}$ and $[H_t^P]_{t=0}^{\infty}$, such that g_t^N and H_t^P admit strictly positive limits. If i) the probability that a new product starts out non-automated is bounded below away from zero and ii) the intensity at which non-automated firms are automated is bounded above and below away from zero, then any limit of G_t must have $0 < G_\infty < 1$.*

Proof. See Appendix 8.2.2. □

In other words, as long as new non-automated products are continuously introduced (and stay non-automated for a non negligible time period), there will always be a share of non-automated products. These provide employment opportunities for low-skill workers which limits the relative losses of low-skill workers compared to high-skill workers (in that their wages grow according to (14) instead of (15)). This is endogenously what will happen in the full dynamic model that we now turn to.

[10]This provides one possible micro-foundation for the "Android Experiment" in Brynjolfsson and McAffee (2014) where an android is invented which can perform any task a human worker can do. Proposition 2 demonstrates that even if (asymptotically) this state is reached, low-skill workers will get the opportunity cost of such an android which in general will not tend to zero. Appendix 7.3 demonstrates that a necessary (though not sufficient) condition for low-skill wages to approach zero is that the cost of machines/androids falls faster than the consumption good.

Note that, the intuition given by the combination of Lemma 2 and Part A of Proposition 2 does not rely on new products being born identical to older products. In a model where new products are born more productive, the growth rate of high-skill wages and low-skill wages will obey equations (13) and (14), as long as the automation intensity is bounded and the economy grows at a positive but finite rate.

3 Endogenous innovation

We now model automation and horizontal innovation as a the result of intentional investment, which allows us to look at the impact of wages on technological change. Section 3.1-3.5 characterizes the model and its solution, section 3.6 relates it to the historical experience and section 3.7 is key as it studies the interaction between the two innovation processes.

3.1 Modeling innovation

We assume that automation results from a risky investment: a non-automated firm which hires $h_t^A(i)$ high-skill workers in automation research, becomes automated according to a Poisson process with rate $\eta G_t^{\widetilde{\kappa}}\left(N_t h_t^A(i)\right)^{\kappa}$. Once a firm is automated it remains so forever. $\eta > 0$ denotes the productivity of the automation technology, $\kappa \in (0,1)$ measures the concavity of the automation technology, $G_t^{\widetilde{\kappa}}$, $\widetilde{\kappa} \in [0,\kappa]$, represents possible knowledge spillovers from the share of automated products, and N_t represents knowledge spillovers from the total number of intermediate inputs. The spillovers in N_t are necessary to ensure that both automation and horizontal innovation can take place in the long-run.[11] Our set-up can be interpreted in two ways. From one standpoint, machines are intermediate input-specific and each producer needs to invent his own machine, which, once invented, is produced with the same technology as the consumption good.[12] From a sec-

[11] The role of these spillovers is to compensate for the mechanical reduction in the amount of resources for automation that are available for each product when the number of product increases. With less spillovers (that is if the process depended on $N_t^q h_t^A$ with $q < 1$) automation would disappear as the amount of effective resources per firm available for automation ($N_t^q H/N_t$) would become arbitrarily small. With more spillovers ($q > 1$), the reverse occurs and firms could asymptotically get automated instantaneously. Furthermore, these spillovers can be micro-funded as follows: let there be a mass 1 of firms with N_t products (instead of assuming that each individual i is a distinct firm), then this functional form means that when a firm hires a mass $\widetilde{H}_t^A(i)$ of high-skill workers in automation each of its non-automated products gets independently automated with a Poisson rate of $\eta G_t^{\widetilde{\kappa}}\left(\widetilde{H}_t^A(i)/\left(1-G_t\right)\right)^{\kappa}$.

[12] Alternatively, machine-i may be invented by an outside firm and then sold to the intermediate input producer. The rents from automation would then be divided between the intermediate input

ond standpoint, machines are produced by the final good sector, and each intermediate input producer must spend resources in adapting machines to his product line so as to make them substitutable with low-skill workers in a new set of tasks.

Horizontal innovation occurs in a standard manner. New intermediate inputs are developed by high-skill workers according to a linear technology with productivity γN_t, where $\gamma > 0$ is a productivity parameter.[13] With H_t^D high-skill workers pursuing horizontal innovation, the mass of intermediate inputs evolves according to:[14]

$$\dot{N}_t = \gamma N_t H_t^D.$$

We assume that firms do not exist before their product is created. Coupled with our assumption that automation follows a continuous Poisson process, new products must then be born non-automated. This feature of the model is motivated by the idea that when a task is new and unfamiliar, the flexibility and outside experiences of workers allow them to solve unforeseen problems. As the task becomes routine and potentially codefiable a machine (or an algorithm) can perform it (as argued by Autor, 2013). In reality, some new tasks may be sufficiently close to older ones that no additional investment would be required to automate them immediately. Our results carry through if only a share of the new products are born non-automated and section 3.8 discusses an alternative set-up where automation is only undertaken at the entry stage.[15]

Therefore the rate and direction of innovation will depend on the equilibrium allocation of high-skill workers between production, automation and horizontal innovation. Defining the total mass of high-skill workers working in automation as $H_t^A \equiv \int_0^{N_t} h_t^A(i)di$,

producer and the machine producer. Except for a constant representing the bargaining power of each party, it would not affect any of our results. Yet another alternative would be to have entrants undertaking automation and potentially displacing the original firm. This would not qualitatively affect the equilibrium as long as the incumbent has a positive probability of becoming automated.

[13]As is common in the growth literature, in this set-up each firm is assumed to produce a good different from the others. Horizontal innovation, however, does not aim to represent the creation of new firms but the creation of new goods or services.

[14]The main results of this paper do not depend on the fact that firms are born non-automated. Appendix 8.6 presents a setting in which when a firm is born (and only when it is born), its owner can make it automated with probability $\min(\eta(N_t h_t^A)^\kappa, 1)$ by hiring h_t^A high-skill workers in automation. The transition of the economy is qualitatively identical. What is crucial is that higher low-skill wages increase the incentive to automate, not at what stage this automation takes place.

[15]The model predicts that the ratio of high-skill to low-skill labor *in production* is higher for automated than non-automated firms. However, this does not necessarily mean that automated firms have a higher ratio of high-skill to low-skill labor overall, since non-automated firms also hire high-skill workers for the purpose of automating. In particular, new firms do not always have a higher ratio of low to high-skill workers (and at a the time of its birth a new firm only relies on high-skill workers).

we get that high-skill labor market clearing leads to

$$H_t^A + H_t^D + H_t^P = H. \tag{16}$$

3.2 Innovation allocation

We denote by V_t^A the value of an automated firm, by r_t the economy-wide interest rate and by $\pi_t^A \equiv \pi(w_t, v_t, 1)$ the profits at time t of an automated firm. The asset pricing equation for an automated firm is then given by

$$r_t V_t^A = \pi_t^A + \dot{V}_t^A. \tag{17}$$

This equation states that the required return on holding an automated firm, V_t^A, must equal the instantaneous profits plus appreciation. An automated firm only maximizes instantaneous profits and has no intertemporal investment decisions to make.

A non-automated firm has to decide how much to invest in automation. Denoting by V_t^N the value of a non-automated firm, we get the corresponding asset pricing equation:

$$r_t V_t^N = \pi_t^N + \eta G_t^{\widetilde{\kappa}} \left(N_t h_t^A\right)^{\kappa} \left(V_t^A - V_t^N\right) - v_t h_t^A + \dot{V}_t^N, \tag{18}$$

where $\pi_t^N \equiv \pi(w_t, v_t, 0)$ and h_t^A is the mass of high-skill workers in automation research hired by a single non-automated firm (by symmetry $H_t^A = (1-G_t)N_t h_t^A$). This equation has an analogous interpretation to equation (17), except that profits are augmented by the instantaneous expected gain from innovation $\eta G_t^{\widetilde{\kappa}} \left(N_t h_t^A\right)^{\kappa} \left(V_t^A - V_t^N\right)$ net of expenditure on automation research, $v_t h_t^A$. This gives the first order condition:

$$\kappa \eta G_t^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa - 1} \left(V_t^A - V_t^N\right) = v_t, \tag{19}$$

which must hold at all points in time. The mass of high-skill workers hired in automation increases with the difference in value between automated and non-automated firms, and as such is increasing in current and future low-skill wages—all else equal.

Since non-automated firms get automated at Poisson rate $\eta G_t^{\widetilde{\kappa}} \left(N_t h_t^A\right)^{\kappa}$, and since new firms are born non-automated, the share of automated firms obeys:

$$\dot{G}_t = \eta G_t^{\widetilde{\kappa}} \left(N_t h_t^A\right)^{\kappa} (1 - G_t) - G_t g_t^N, \tag{20}$$

Free-entry in horizontal innovation guarantees that the value of creating a new firm cannot be greater than its opportunity cost:

$$\gamma N_t V_t^N \leq v_t, \tag{21}$$

with equality whenever there is strictly positive horizontal innovation ($\dot{N}_t > 0$).

The low-skill and high-skill representative households' problems are standard and lead to Euler equations which in combination give

$$\dot{C}_t / C_t = (r_t - \rho) / \theta, \tag{22}$$

with a transversality condition requiring that the present value of all time-t assets in the economy (the aggregate value of all firms) is asymptotically zero:

$$lim_{t\to\infty} \left(exp\left(- \int_0^t r_s ds \right) N_t \left((1 - G_t) V_t^N + G_t V_t^A \right) \right) = 0.$$

3.3 Equilibrium characterization

We define a feasible allocation and an equilibrium as follows:

Definition 1. A feasible allocation is defined by time paths of stock of products and share of those that are automated, $[N_t, G_t]_{t=0}^{\infty}$, time paths of use of low-skill labor, high-skill labor, and machines in the production of intermediate inputs $[l_t(i), h_t(i), x_t(i)]_{i\in[0,N_t],t=0}^{\infty}$, a time path of intermediate inputs production $[y_t(i)]_{i\in[0,N_t],t=0}^{\infty}$, time paths of high-skill workers engaged in automation $[h_t^A(i)]_{i\in[0,N_t],t=0}^{\infty}$, and in horizontal innovation $[H_t^D]_{t=0}^{\infty}$, time paths of final good production and consumption levels $[Y_t, C_t]_{t=0}^{\infty}$ such that factor markets clear ((16) holds) and good market clears ((12) holds).

Definition 2. An equilibrium is a feasible allocation, a time path of intermediate input prices $[p_t(i)]_{i\in[0,N_t],t=0}^{\infty}$, a time path for low-skill wages, high-skill wages, the interest rate and the value of non-automated and automated firms $[w_t, v_t, r_t, V_t^N, V_t^A]_{t=0}^{\infty}$ such that $[y_t(i)]_{i\in[0,N_t],t=0}^{\infty}$ maximizes final good producer profits, $[p_t(i), l_t(i), h_t(i), x_t(i)]_{i\in[0,N_t],t=0}^{\infty}$ maximize intermediate inputs producers' profits, $[h_t^A(i)]_{i\in[0,N_t],t=0}^{\infty}$ maximizes the value of non-automated firms, $[H_t^D]_{t=0}^{\infty}$ is determined by free entry, $[C_t]_{t=0}^{\infty}$ is consistent with consumer optimization and the transversality condition is satisfied.

We transform the system by introducing new variables for which the system of differential equations admits a steady state. Specifically, we introduce $n_t \equiv N_t^{-\beta/[(1-\beta)(1+\beta(\sigma-1))]}$

and $\omega_t \equiv w_t^{\beta(1-\sigma)}$ which both tend towards 0 as N_t and w_t tend towards infinity.[16] We define the normalized mass of high-skill workers in automation ($\hat{h}_t^A \equiv N_t h_t^A$), normalized high-skill wages and consumption ($\hat{v}_t = v_t N_t^{-\psi}$ and $\hat{c}_t = c_t N_t^{-\psi}$), where $\psi \equiv ((1-\beta)(\sigma-1))^{-1}$ (ψ is equal to the asymptotic elasticity of GDP with respect to N_t), and the variable $\chi_t \equiv \hat{c}_t^{\,\theta}/\hat{v}_t$ ($\chi_t$is related to the mass of high-skill workers in production and therefore, given $\hat{h}_t^A$, to H_t^D and the growth rate of N_t). With positive entry in the creation of new products at all points in time, the equilibrium can then be characterized by a system of differential equations with two state variables n_t, G_t, two control variables, $\hat{h}_t^A, \chi_t$ and an auxiliary equation defining ω_t (see Appendix 7.1 for the derivation, in particular the system is given by equations (27), (28), (30) and (31) with auxiliary variables defined in (35), (37), (38) and (39)). We then get:

Proposition 3. *Assume that*

$$\rho\left(\frac{1}{\eta\kappa^{\kappa}(1-\kappa)^{1-\kappa}}\left(\frac{\rho}{\gamma}\right)^{1-\kappa}+\frac{1}{\gamma}\right)<\psi H, \tag{23}$$

then the system of differential equations admits a steady state $(n^*, G^*, \hat{h}^{A*}, \chi^*)$ *with* $n^* = 0$*,* $0 < G^* < 1$ *and positive growth* $\left(g^N\right)^* > 0$*.*

Proof. See Appendix 8.3.1. □

We will refer to the steady state $(n^*, G^*, \hat{h}^{A*}, \chi^*)$ as as an asymptotic steady state for our original system of differential equations. In addition, the assumption that $\theta \geq 1$ ensures that the transversality condition always holds.[17] For the rest of the paper we restrict attention to parameters such that there exists a unique saddle-path stable steady state $(n^*, G^*, \hat{h}^{A*}, \chi^*)$ with $n^* = 0$, $G^* > 0$. Then, for an initial pair $(N_0, G_0) \in (0, \infty) \times [0, 1]$ sufficiently close to the asymptotic steady state, the model features a unique equilibrium converging towards the asymptotic steady state.[18]

[16] Because the original system only admits an asymptotic steady state, one cannot eliminate the state variable N_t from the system. We introduce n_t to be able to define a proper steady state.

[17] To see the intuition behind equation (23), consider the case in which the efficiency of the automation technology η is arbitrarily large, such that the model is arbitrarily close to a Romer model where all firms are automated. Then equation (23) becomes $\rho/\gamma < \psi H$, which mirrors the classical condition for positive growth in a Romer model with linear innovation technology. With a smaller η the present value of a new product is reduced such that the corresponding condition is more stringent

[18] Multiple asymptotic steady states with $G^* > 0$ are technically possible but are not likely for reasonable parameter values (see Appendix 8.3.2). In addition, with two state variables (n_t and G_t) saddle path stability requires exactly two eigenvalues with positive real parts. In our numerical investigation, for all parameter combinations which satisfy the previous restrictions, this condition was always met.

In line with Lemma 2, the steady state features $G^* \in (0,1)$ as the automation intensity is positive but bounded. Therefore Proposition 2, part A applies: asymptotically high-skill wages grow faster than low-skill wages but the introduction of new non-automated products limits the ratio between the two growth rates.

3.4 Innovation incentives along the transitional path

A distinctive feature of this economy is that the path of technological change itself will be unbalanced through the transitional dynamics. In the following we think of the transitional path of the economy as going through three phases: a first phase where the incentive to automate is very low and the economy behaves close to a Romer model, a second phase in which automation pushes up G and the skill-premium condition of figure 1 pivots counter-clockwise and a final third phase where the economy approaches the steady state described by the previous section (a formal proof of the results in this section can be found in Appendix 8.3.3).

To see this, we combine (17) (18) and (19) to write the difference in value between an automated and a non-automated firm as:

$$r_t(V_t^A - V_t^N) = \pi_t^A - \pi_t^N - \frac{1-\kappa}{\kappa} v_t h_t^A + (\dot{V}_t^A - \dot{V}_t^N).$$

Integrating over this equation (and using the transversality condition), we obtain that the difference in value between an automated and a non-automated firm is given by the discounted flow of the difference in profits adjusted for the cost of automation and the probability of getting automated:

$$V_t^A - V_t^N = \int_t^\infty \exp\left(-\int_t^\tau r_u du\right)\left(\pi_\tau^A - \pi_\tau^N - \frac{1-\kappa}{\kappa} v_\tau h_\tau^A\right) d\tau. \tag{24}$$

The rate of automation depends on the normalized mass of high-skill workers in automation ($\hat{h}_t^A = N_t h_t^A$), which following (19) depends on the ratio between the gain in firm value from automation and the high-skill wage divided by the number of products:

$$\hat{h}_t^A = \left(\kappa \eta G_t^{\widetilde{\kappa}} \frac{V_t^A - V_t^N}{v_t / N_t}\right)^{1/(1-\kappa)}. \tag{25}$$

Crucially as the number of products in the economy increases, the right-hand side of this expression will change value drastically.

First Phase. High-skill wages v_t and aggregate profits are of the same order (both are proportional to output). When the number of products N_t is low, wages, including low-skill wages are low and therefore automated and non-automated firms have similar profits. As a result aggregate profits are close to $N_t \pi_t^N$ so that π_t^N and v_t / N_t are of the same order (and grow at similar rates). Recalling that (7) gives $\pi_t^A - \pi_t^N = \left(1 + \varphi w_t^{\varepsilon-1}\right)^{\mu} \pi_t^N$, we get that as long as w_t is small relative to $\widetilde{\varphi}^{-1}$, then $\left(\pi_t^A - \pi_t^N\right) / \left(v_t / N_t\right)$ must be small too.[19] With a positive discount rate, this implies that $\left(V_t^A - V_t^N\right) / \left(v_t / N_t\right)$ and therefore $\hat{h}_t^A$ must also be small (see Appendix 8.3.3 for a proof). Hence the economy initially experiences little automation.

Therefore, for sufficiently low initial value of N_t the behavior of the economy is close to that of a Romer model with a Cobb-Douglas production function between low-skill and high-skill labor. Economic growth is driven by horizontal innovation and the skill premium and the factor shares are nearly constant. Naturally, if G_t is not initially low, it must depreciate during this period following equation (20). This corresponds to what we label as the first phase of the economy.

Second Phase. As horizontal innovation continuously increases low-skill wages (for low N_t, w_t grows at the rate $g_t^N / (\sigma - 1)$) the approximation derived on the basis that w_t is small relative to $\widetilde{\varphi}^{-1}$ becomes progressively worse. The gain from automating from $\left(V_t^A - V_t^N\right) / \left(v_t / N_t\right)$ increases and ceases to be a low-number.[20] Without the externality in the automation technology ($\widetilde{\kappa} = 0$), it is then direct from (25) that the mass of high-skill workers devoted to automation becomes significantly different from 0, so that the Poisson rate of automation $\eta \left(N_t h_t^A\right)^{\kappa}$ increases and so does the share of automated products G_t. For $\widetilde{\kappa} > 0$, the depreciation in the share of automated products during the first phase might gradually makes the automation technology less effective which can delay or even potentially prevent the take-off of automation.[21]

In line with Proposition 1, both the increase in G_t but also the increase in the number of products (now with G_t substantially higher than 0) lead to an increase in the skill premium. We will label this time period where the share of automated products in the economy increases sharply the second phase (the transition between phases is smooth

[19]More specifically one finds that $\frac{\pi_t^A - \pi_t^N}{v_t / N_t} = \psi H_t^P \frac{\left(\left(1+\varphi w_t^{\epsilon-1}\right)^{\mu} - 1\right)}{1 + G_t\left(\left(1+\varphi w_t^{\epsilon-1}\right)^{\mu} - 1\right)}$, which is small if $\varphi w_t^{\epsilon-1}$ is small and increasing in w_t(and therefore in N_t) for given G_t and H_t^P.

[20]When G_t is low, we still have that $N_t \pi_t^N$ and v_t are of the same order and as a result the ratio $\left(V_t^A - V_t^N\right) / \left(v_t / N_t\right)$ grows like $\left(\pi_t^A - \pi_t^N\right) / \pi_t^N = \left(1 + \varphi w_t^{\varepsilon-1}\right)^{\mu} - 1$.

[21]Automation does still take off if either G_0 and N_0 are not too low or, for any values of $N_0, G_0 > 0$ whenever $1 - \kappa - \widetilde{\kappa} > 0$—see Appendix 8.3.4. Finally, if we were to assume instead that the automation technology is given by $\min\left\{\eta G_t^{\widetilde{\kappa}}, \underline{\eta}\right\} \left(N_t h_t^A\right)^{\kappa}$, then automation would always take off.

and therefore the exact limits are arbitrary). Arguably, this time period is the one where our model differs the most from the rest of the literature.

Third Phase. With a share of automated products significantly different from 0, aggregate profits and the profits of an automated firm multiplied by the number of products ($N_t \pi_t^A$) must be of the same order. Recalling that high-skill wages and aggregate profits are both proportional to output, this ensures that the ratio $\left(V_t^A - V_t^N\right) / (v_t/N_t)$ remains bounded. As a result the normalized mass of high-skill workers in automation research ($N_t h_t^A$) also stays bounded (see (25)). In line with Lemma 2 a bounded Poisson rate of automation ensures that the share of automated products stabilizes below 1. Therefore, the economy will experience a third phase where the share of automated products is approximately constant. Following Proposition 1, both high-skill and low-skill wages grow but high-skill wages grow at a higher rate. Though in Phase 3 the share parameters of the nested CES function are constant, the model continues to differ from a generic capital deepening model in that long-run growth is endogenized and depends on its interaction with automation (in particular, see Proposition 4 below).

3.5 An illustration of the transitional dynamics

In order to illustrate the previous results and to further analyze the behavior of our economy under various parametric assumptions, we now turn to numerical simulations.[22] The following section will then relate our theoretical results to the historical experience of the US economy. Table 1 presents our baseline parameters. Section 4 employs Bayesian techniques to estimate the parameters, but the focus of this section is theoretical and we simply choose "reasonable" parameters (see Appendix 7.2.4 for a systematic exploration of the parameter space). As our goal is to characterize the evolution of an economy which transitions from automation playing a small to a central role, we choose an initially low level of automation ($G_0 = 0.001$) and an initial mass of intermediate inputs small enough to ensure that the real wage is initially low relative to the productivity of machines. This ensures that the economy will start in the 'first phase' described above (with a higher (N_0, G_0) the economy may directly start in the second or third phases).

Baseline Parameters. The time unit is 1 year. Total stock of labor is 1 and we set $L = 2/3$ and $\beta = 2/3$ such that absent automation and if all high-skill workers were in production the skill premium would be 1. The initial mass of products is $N_0 = 1$ and

[22] We employ the so-called "relaxation" algorithm for solving systems of discretized differential equations (Trimborn, Koch and Steger, 2008). See Appendix 8.4 for details.

Table 1: Baseline Parameter Specification

σ	ϵ	β	H	L	θ	η	κ	$\tilde{\varphi}$	ρ	$\widetilde{\kappa}$	γ
3	4	2/3	1/3	2/3	2	0.2	0.5	0.25	0.02	0	0.3

the productivity parameter for machines is $\tilde{\varphi} = 0.25$, which ensures that at $t = 0$, the cost advantage of automated firms is very small (their profits are 0.004% higher). We set $\sigma = 3$ to capture an initial labor share close to 2/3. The elasticity of substitution between machines and low-skill workers in automated firms is $\epsilon = 4$. The innovation parameters (γ, η, κ) are chosen such that GDP growth is close to 2% both initially and asymptotically, and we first consider the case where there is no externality from the share of automated products in the automation technology, $\widetilde{\kappa} = 0$—hereafter, we will refer to this externality as the externality in automation technology (although there is also an externality from the total mass of products). The parameters ρ and θ are chosen such that the interest rate is around 6% (at the beginning and at the end of the transition).

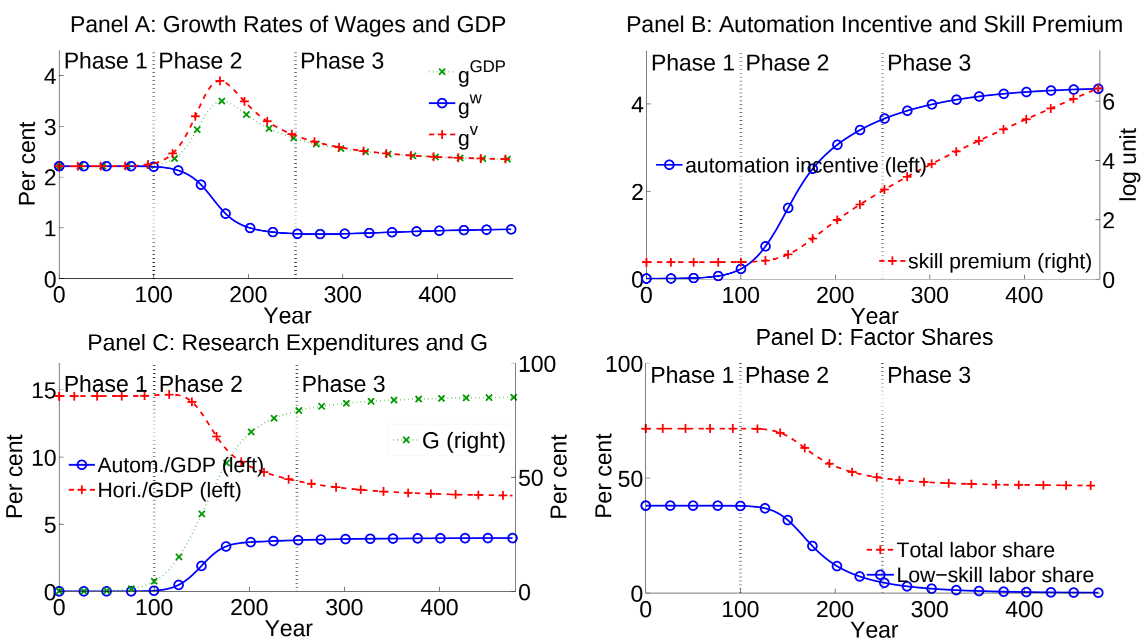

Figure 2: Transitional Dynamics for baseline parameters. Panel A shows growth rates for GDP, low-skill wages (w) and high-skill wages (v), Panel B the incentive to automate, $\left(V_t^A - V_t^N\right) / \left(v_t/N_t\right)$, and the skill premium, Panel C the total spending on horizontal innovation and automation as well as the share of automated products (G), and Panel D the wage share of GDP for total wages and low-skill wages.

Figure 2 plots the evolution of the economy. Based on the behavior of G_t (Panel C) we delimit Phase 1 as corresponding to the first 100 years and Phase 2 as the period between year 100 and year 250.

Innovation. As previously derived and as shown in Panel C firms initially spend very little on automation and the share of automated firms, G_t, stays initially very close to 0. This occurs because with a low initial level of N_t, low-skill wages are low, so that the gains from automation, $V_t^A - V_t^N$, are very low relative to the high-skill wage normalized by the number of products (v_t/N_t)—as shown in Panel B. With growing low-skill wages, the incentive to automate picks up a bit before year 100. Then, the economy enters the second phase as automation expenses sharply increase (up to 4% of GDP), leading to an increase in the share of automated products G_t. Despite a high level of expenditures in automation, the share of automated products eventually stabilizes in the third phase below 1, as the constant arrival of new, non-automated products depreciates it.

As shown in Panel C, spending on horizontal innovation as a share of GDP declines during Phase 2 and ends up being lower in Phase 3 than Phase 1. Intuitively, horizontal innovation becomes less interesting because new (non-automated) products will have to compete with increasingly productive automated firms and therefore get a smaller initial market share. Though this is not a general feature of our model, such possibility distinguishes our paper from the 'Habbakuk hypothesis' literature: high level of wages may encourage some innovation (automation), but it may also discourage other forms of innovation, leading to an ambiguous overall impact on growth.

Panel A shows that GDP growth is the highest in the middle of Phase 2 and roughly the same in Phases 1 and 3: the rate of horizontal innovation is lower in Phase 3 but this is compensated by the elasticity of GDP wrt. N_t being higher (at $1/(\sigma-1)$ instead of $1/[(\sigma-1)(1-\beta)]$). As a result, the phase of intense automation (when the share of automated products increases) is associated with a temporary boost of growth.

Wages. In the first phase, and referring to figure 1, the skill-premium conditions stays close to the straight dotted line with slope $\frac{1-\beta}{\beta}\frac{L}{H^P}$ associated with a Cobb-Douglas production. Continuous horizontal innovation pushes the productivity condition towards the North East so that both wages grow at around 2% (Panel A).

As rising low-skill wages trigger the second phase, the skill-premium condition pivots counter-clockwise and bends upwards increasing the growth rate of high-skill wages to almost 4% and suppressing the growth rate of low-skill wages to around 1%. (since there are no financial constraints, the two types share a common consumption growth rate throughout, see Appendix 7.2.1). For our specific parameter choice (satisfying the conditions of Proposition 1 B.ii)), increases in G_t have a negative impact on w_t throughout the transition, but it is sufficiently slow relative to the increase in N_t that low-skill wages grow at a positive rate throughout—which section 3.7 demonstrates need not be the case. It is precisely this movement of the skill-premium curve that an alternative model with constant G (i.e. one where the fraction of tasks that can be performed with machines is constant) or a capital deepening model could not reproduce, and consequently such a model would not feature labor-saving innovation. In addition to the effects of changing G_t and N_t, changes in the mass of high-skill workers in production, H_t^P, affect the skill premium, but these effects are quantitatively dominated. In the third phase, the skill-premium condition no longer moves, but the continuous rise in the productivity condition continuous to increase the skill-premium.

Capital and labor shares. Panel D of figure 2 plots the labor share and the low-skill labor share. To understand their evolution, first note that $GDP = Y - X + v(H^D + H^A)$, as GDP includes R&D investments done by high-skill labor, but not intermediate inputs, X. Since machines are not part of the capital stock in this baseline version of the model (see Appendix 7.4 for an alternative specification), capital income corresponds to aggregate profits. Due to a constant markup, these profits are a constant share of output. During the first phase, X is low and $v(H^D + H^A)$ are roughly a constant share of output, implying a nearly constant capital share. With low-skill and high-skill wages growing at the same rate, the low-skill share is also constant during this period.

With the advent of automation in Phase 2, the increased use of intermediate inputs implies a decreasing GDP/Y and profits and thereby the capital share becomes a growing share of GDP. Working contrary to this is the increase in innovation as only high-skill workers work in innovation, but the net effect is an increase in the capital share. For different parameter values, the drop in the labor share can be delayed relative to the rise in the skill premium (see Appendix 7.2.2). As the growth rate of low-skill wages starts declining and falls to a level permanently lower than that of GDP, the low-skill labor share declines eventually to approach zero. The high-skill wage share, however, asymptotically grows at the rate of GDP and stabilizes at a higher value than in Phases 1 and 2. The ratio of wealth to GDP also increases during Phase 2 and asymptotes a constant in Phase 3 (see Appendix 7.2.1).

Elasticity of substitution. At the aggregate level, our model boils down to a nested CES production function (see equation (3)), and Phase 2 corresponds to a period where the share parameter of the composite which features substitutability between machines and low-skill labor, G_t, rises. This change in the share parameter is microfounded and receives a very natural interpretation, which is precisely the advantage of a task framework. In contrast, the academic debate on income distribution often focuses on the value of the elasticity of substitution between different factors. Here the value of the aggregate elasticity of substitution does not play the central role. In fact, the Morishima's elasticity of substitution between low-skill labor and machines (or machines and low-skill labor) actually declines in Phase 2 from a value close to ϵ to a value close to $1 + \beta(\sigma - 1)$ (see Appendix 8.3.5).

Growth decomposition. Figure 3 performs a growth decomposition exercise for low-skill and high-skill wages by separately computing the instantaneous contribution of each type of innovation. We do so by performing the following thought experiment: at a given instant t, for given allocation of factors, suppose that all innovations of a given type fail. By how much would the growth rates of w and v change? This exercise is complementary to the one performed in Figure 1 which focuses on the impact of technological levels instead of innovations.[23] In Phase 1, there is little automation, so wage growth for both skill-groups is driven almost entirely by horizontal innovation. In Phase 2, automation sets in. Low-skill labor is then continuously reallocated from existing products which get automated, to new, not yet automated, products. The immediate impact of automation on low-skill wages is negative, while horizontal innovation has a positive impact, as it both increases the range of available products and decreases the share of automated products. The figure also shows that automation plays an increasing role in explaining the growth rate of high-skill wages, while the contribution of horizontal innovation declines. This is because new products capture a smaller and smaller share of the market and therefore do not contribute much to the demand for high-skill labor. Consequently, automation is skill-biased while horizontal innovation is unskilled-biased.[24] We stress that this growth decomposition is for changes in the rate

[23] More specifically we can write $w_t = f(N_t, G_t, H_t^P)$, using equations (10) and (11). Differentiating with respect to time and using equation (28) gives:

$$g_t^w = \left(\frac{N_t}{w_t} \frac{\partial f}{\partial N} - \frac{G_t}{w_t} \frac{\partial f}{\partial G} \right) \gamma H_t^D + \frac{1}{w_t} \frac{\partial f}{\partial G} \eta G_t^{\tilde{\kappa}} (1 - G_t) (\hat{h}_t^A)^{\kappa} + \frac{1}{w_t} \frac{\partial f}{\partial H^P} \dot{H}_t^P .$$

Figure 3 plots the first two terms as the growth impact of expenses in horizontal innovation and automation, respectively. The third term ends up being negligible. We perform a similar decomposition for v_t.

[24] Proposition 1 shows analytically that automation is skilled-biased and that an increase in N at a given $G > 0$ is also skilled-biased. Horizontal innovation corresponds to an increase in the number of non-automated products, that is an increase in N but keeping GN constant. One can show analytically

of automation and horizontal innovation at a given point in time. This should not be interpreted as "automation being harmful" to low-skill workers in general. In fact, as we demonstrate in Section 3.7, an increase in the effectiveness of the automation technology, η, will have positive long-term consequences. A decomposition of g_t^{GDP} would look similar to the decomposition of g_t^v: while growth is initially almost entirely driven by horizontal innovation, automation becomes increasingly important in explaining growth.[25]

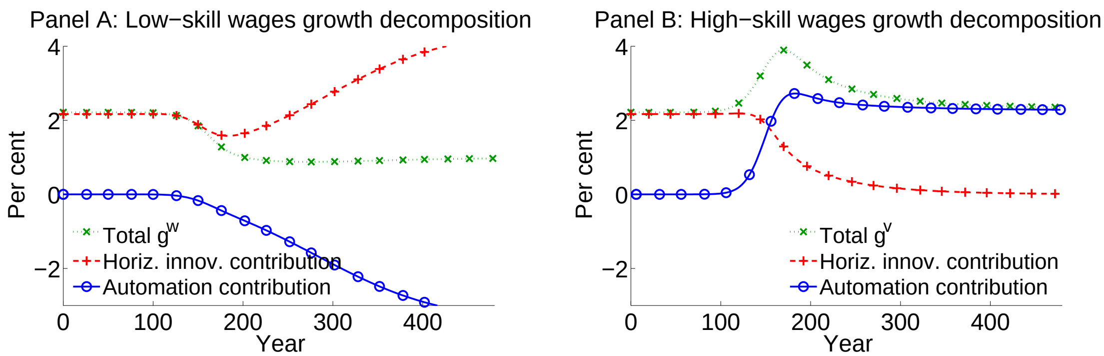

Figure 3: Growth decomposition. Panel A: The growth rate of low-skill wages and the instantaneous contribution from horizontal innovation and automation, respectively. Panel B is analogous for high-skill wages. See text for details.

3.6 Comparison with the historical experience

We now relate the qualitative lessons of our model to historical experience. Although, when undertaking our quantitative analysis in section 4, we will focus on the last 40 years, we believe that the forces at work in our model were relevant much before.

Secular increase in the relative skill demand. Our model predicts a continuous increase in the skill premium from Phase 2. It is a well established fact that the college premium (considered to be a good proxy for the skill premium over that time period) has been steadily increasing in the United States since the 1980s. Moving back in time, the skill premium experienced periods of decline (such as the 1970s) but these can be accounted for by exogenous changes in the relative supply of skills which our models does not capture. Factoring in the evolution on the supply side, Goldin and Katz (2008) show that technological change has been skill-biased throughout the 20^{th} century. Even before, Katz and Margo (2014) argue that the relative demand for highly skilled workers (in

that horizontal innovation is unskilled-biased when w is high enough (which is obtained for $\epsilon < \infty$ and N large enough), but might be skill biased for low N.

[25]This is about instantaneous growth, as shown in Proposition 2, long-run growth is ultimately determined by the endogenous rate of horizontal innovation.

professional, technical and managerial occupations) has increased steadily from perhaps as early as 1820 to the present.

This contrasts our paper with most of the growth literature which features a balanced growth path and therefore does not have permanently increasing labor inequality. For instance, in Acemoglu (1998), low-skill and high-skill workers are imperfect substitutes in production. Yet, since the low-skill augmenting technology and the high-skill augmenting technology grow at the same rate asymptotically, the relative stocks of effective units of low-skill and high-skill labor is constant, leading to a constant relative wage.

Furthermore, there is no simple one-to-one link between automation spending and rising inequality in our model. Here, automation spending is higher in Phase 3 than in Phase 2 (Panel C in figure 2), yet the growth in the skill premium is slower. Card and DiNardo (2002) argue that inequality rising the most in the early to mid 1980s, and technological change continuing since squares poorly with the predictions of a framework based on skill-biased technological change. This, in fact, is in line with our model.[26]

Of course, the definition of who is 'high' skill and who is 'low' skill matters. Hence, the mechanization of the 19^{th} century, which replaced skilled artisans, or the computerization of the last 30 years, did not aim at replacing the most unskilled workers. Therefore our model provides a good account of the historical experience only if the definition of 'high-skill' is restrictive. The next section, which introduces a group of middle-skill workers, helps us account for such events.

Capital and labor shares. Our model also predicts a slow drop in the labor share during most of Phase 2 (and a rise in the capital to income ratio). This is consistent with Karabarbounis and Neiman (2013) who find a global reduction of 5 percentage points in labor's share of corporate gross value added over the past 35 years, and with Elsby, Hobijn and Sahin (2013) who find similar results for the United States.

However, the capital share of income and the wealth to income ratio have followed a U-curve in the 20^{th} century (Piketty and Zucman, 2014 and Piketty, 2014). Although a small temporary decline in the capital share at the beginning of Phase 2 can be accounted for by the model (see Appendix 7.2.2), such large movements cannot. The early decline in the capital share is at least partly due to the two World Wars, changes in the tax system and the structural shift away from the agricultural sector, which this model does not capture. The latter, could be captured with a nested structure with an elasticity of substitution between broad sectors of less than 1. If these sectors differ in how easy

[26]Intuitively the elasticity of the skill-premium with respect to the skill-bias of technology is not constant in our model, contrary to a CES framework with factor-augmenting technologies.

they are to automate, then intense automation will happen sequentially. As this happens spending shares in non-automated sectors would increase (as in Acemoglu and Guerrieri, 2008) securing a higher growth rate for low-skill wages. This would replicate the broad features of an economy switching from agriculture, to manufacturing, and then services.[27]

3.7 Interaction between horizontal innovation and automation

We now investigate the interaction between horizontal innovation and automation by changing the innovation parameters. This allows us to explore more fully the richness of a model with endogenous technological change and to distinguish it from alternative models. Appendix 7.2.4, carries out a more systematic comparative statics exercise.

Declining low-skill wages. Empirical evidence suggests that low-skill wages have been stagnating and perhaps even declining in recent periods. Because our model features a labor saving innovation, it can accommodate declining low-skill wages in Phase 2. This could not happen if the share of automated products were fixed, or in a capital deepening model with perfectly elastic capital, and it is not the case in the previous DTC literature either. In this model it happens when the skill-premium pivots sufficiently fast counter-clockwise compared with the movement of the productivity condition in figure 1. We ensure this in figure 4 by setting $\tilde{\kappa} = 0.49$, thereby introducing the externality in automation.[28] Initially G_t is small and the automation technology is quite unproductive. Hence, Phase 2 starts later, even though the ratio $\left(V_t^A - V_t^N\right) / \left(v_t/N_t\right)$ has already significantly risen (Panel B). Yet, Phase 2 is more intense once it gets started, partly because of the sharp increase in the productivity of the automation technology (following the increase in G_t) and partly because low-skill wages are higher. Intense automation puts downward pressure on low-skill wages. At the same time, horizontal innovation drops considerably, both because new firms are less competitive than their automated counterparts, and because the high demand for high-skill workers in automation innovations increases the cost of inventing a new product. This results in a short-lived decline in low-skill wages. Indeed, the decline in w_t (and increase in high-skill wage v_t) lowers the incentive to automate (Panel B), which in return reduces automation. This reflects two general points: First, just as increases in w_t encourage automation; reductions in

[27] In addition, at the product level, an elasticity of substitution lower than 1 between the low-skill/machines aggregate and high-skill workers would secure a higher level for the labor share.

[28] We choose this value for $\widetilde{\kappa}$ instead of 0.5, because in that case there is no horizontal innovation for some time periods (that is (21) holds with a strict inequality). This is not an issue in principle but simulating this case would require a different numerical approach.

w_t discourage it. Second, decreases in wages require two conditions to be met: First, it must be technically possible for low-skill wages to decrease, which here can be done by increases in G_t. Second, it must be privately optimal for agents to choose such a path for G_t. Here this condition can be met by either of two assumptions: one, the externality in automation ($\tilde{\kappa} > 0$) and second the fact that innovation automation has to be paid as an upfront cost instead of a fixed cost every period (and consequently low-skill wages can drop for $\tilde{\kappa} = 0$ — albeit for a small parameter set — see Appendix 7.2.3).

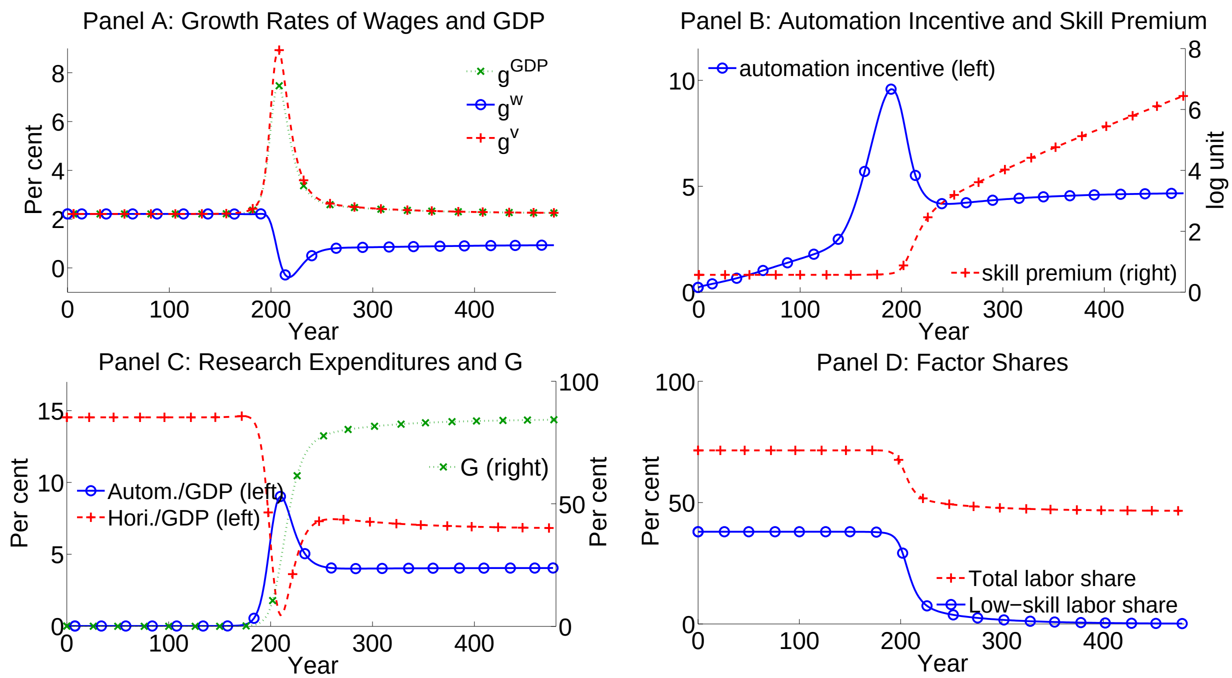

Figure 4: Transitional Dynamics. Note: same as for Figure 2 but with an automation externality of $\tilde{\kappa} = 0.49$.

Innovation parameters. Figure 5 shows the impact (relative to the baseline case) of increasing productivity in the automation technology to $\eta = 0.4$ (from 0.2) and the productivity in the horizontal innovation technology to $\gamma = 0.32$ (instead of 0.3). A higher η initially has no impact during Phase 1, but it moves Phase 2 forward as investing in automation technology is profitable for lower level of low-skill wages. Since automation occurs sooner, the absolute level of low-skill wages drops relative to the baseline case (Panel B), which leads to a fast increase in the skill premium. However, as a higher η means that new firms automate faster, it encourages further horizontal innovation, A faster rate of horizontal innovation implies that the skill premium keeps increasing relative to the baseline, but also that low-skill wages are eventually larger than in the baseline case. Although not explicitly modeled here, this suggests that a policy which would aim at helping low-skill workers by taxing automation might temporarily help low-skill workers, but could have negative long-term consequences. A higher productivity for horizontal innovation, γ, implies that GDP and low-skill wages initially grow faster than in the baseline. Therefore Phase 2 starts sooner, which explains why the skill premium jumps relative to the baseline case before increasing smoothly. The asymptotic results can be derived formally (see Appendix 8.3.6), and we establish the following proposition.

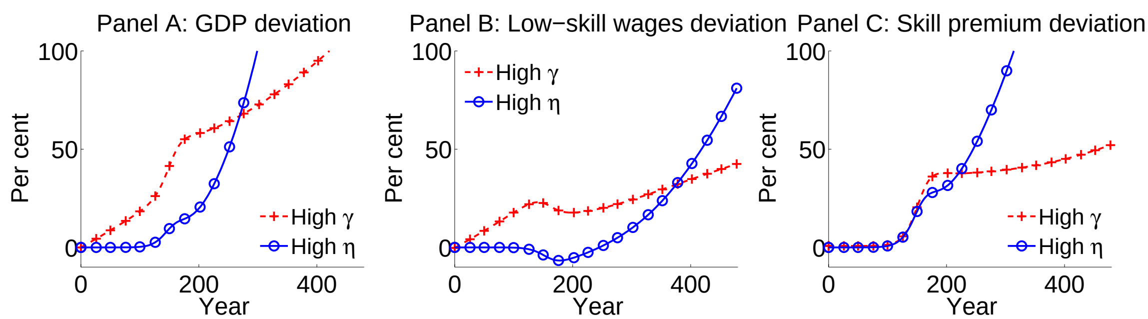

Figure 5: Deviations from baseline model for more productive horizontal innovation technology (γ) and more productive automation technology (η).

Proposition 4. *The asymptotic growth rates of GDP g_∞^{GDP} and of low-skill wages g_∞^w increase in the productivity of automation η and horizontal innovation γ. The asymptotic share of automated products G_∞ decreases in γ.*

Growth and automation. For our simulation in Section 3.5, we chose parameters for which the growth rates in Phases 1 and 3 were close, so that Phase 2 coincided with an increase in GDP growth. This need not be the case, and growth in Phase 3 can be either higher or lower than that of Phase 1. Figure 6 shows a case where Phase 3 growth is substantially lower.[29] As shown in Panel A, growth is a little higher at the beginning of Phase 2 than in Phase 1 but then it continuously decreases in the second part of Phase 2 and is much lower in Phase 3. Intuitively this occurs because horizontal innovation drops sufficiently during Phase 2 as new non-automated firms find it harder to compete with already automated firms. The lack of an acceleration in GDP growth in recent decades has often been advanced in opposition to the hypothesis that a technological revolution

[29] The parameters are identical to the baseline case except for: $\sigma = 2.5$, $\beta = 0.55$, $\eta = 0.1$, $\gamma = 0.23$ and $N_0 = 344.25$, these parameters lower the growth rate of the economy particularly in the asymptotic steady state because automation consumes more resources and is less effective as high-skill workers have a larger factor share in production. N_0 is higher so as to shorten Phase 1 in the graph.

explains the recent increase in the skill-premium (Acemoglu, 2002a). Our model does bear similarities with such a theory (although our technological revolution, automation, can actually be quite progressive and slow); Figure 6 is therefore important in showing that GDP growth need not accelerate. Intuitively, here, the increase in automation research happens at the expense of horizontal innovation.

Conversely, making the automation technology more effective (say by reducing the cost share of high-skill workers, β) could create the opposite pattern of a low initial growth rate followed by a higher eventual growth rate. Such might correspond to the transition through the industrial revolution: Before, the industrial revolution the economy is driven by relatively slow horizontal innovation and there is little incentive to innovate. As wages gradually increase, the incentive to automate rises and the engine of economic growth gradually switches from a relatively inefficient horizontal innovation to a more rapid automation innovation with permanently higher growth as a consequence.

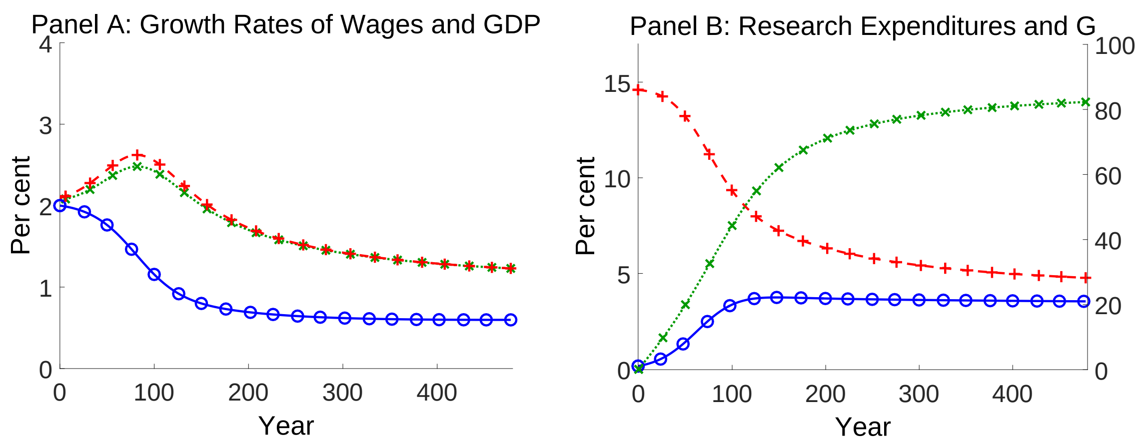

Figure 6: Transitional dynamics with a low growth rate in Phase 2.

3.8 Discussion

Social planner's problem. The social planner's problem is studied in Appendix 8.5. The optimal allocation is qualitatively similar to the equilibrium we described, so that our results are not driven by the market structure we imposed. The social planner correct for four market imperfections: a monopoly distortion, a positive externality in horizontal innovation from the total number of products, a positive externality in the automation technology from the total number of products (the term N_t^{κ}) and a positive externality in the automation technology from the share of automated products when

$\tilde{\kappa} > 0$ (which we referred so far as the "automation externality"). The optimal allocation can be decentralized using lump-sum taxes and the appropriate subsidies to the use of intermediates inputs, horizontal innovation and, if $\tilde{\kappa} > 0$, automation.

Comparison with Acemoglu and Restrepo (2015). In ongoing work, Acemoglu and Restrepo (2015) build a model with automation and the creation of new tasks. Automation plays a similar role in both papers (although in their baseline version, there is only one type of labor), but while in our model new tasks (new products) add up to the stock of existing ones, in their model new tasks are more complex version of existing ones. These new tasks are born non-automated (as in our paper) but with a higher labor productivity (contrary to our paper) which allows them to replace their previous automated vintage. With increasing labor productivity in successive vintages of tasks and under the appropriate assumptions regarding the innovation technology, their model features a stable steady state.[30] The economy self-corrects: if a temporary shock leads to more automation in the short-run, lower wages will reduce incentives to automate, pulling the economy back to its steady state with balanced growth. Yet, their model cannot explain the origin of such a shock and therefore cannot be used to account for trends such as a secular rise in the skill premium. By contrast our model endogenously explains why automation may become more prevalent as an economy develops.[31]

In the rest of the paper, we present two extensions of the baseline model: the first one introduces an endogenous supply response in the skill distribution, and is used to perform a quantitative exercise, the second one includes middle-skill workers and allows the model to account for wage polarization. Besides, Appendix 7.3 presents an extension where the production technology for machines and the consumption good differ, and Appendix 7.4 presents an extension where machines are part of a capital stock.

[30] For this it is crucial that the higher productivity of more complex tasks applies only to labor and not machines. In our model this would be as if in equation (2) there were a labor productivity coefficient $\tau(i)$ in front of $l(i)$ which increases exponentially with i.

[31] Although this is not the focus of our paper, our model also features elements of self-correction in the presence of exogenous shocks. For instance, in the case with no automation externality ($\tilde{\kappa} = 0$), a positive exogenous shock on G_t will be followed by a period where automation is relatively less intense (as the skill premium would have declined), so that eventually the asymptotic share of automated products stays the same.

4 Quantitative Exercise

In this section, we conduct a quantitative exercise to compare empirical trends for the United States for the past 50 years with the predictions of our model using Bayesian techniques. As argued in Goldin and Katz (2008), during this period the relative supply of skilled workers increased dramatically so we let workers switch between skill-types in response to changes in factor rewards.

4.1 An endogenous supply response in the skill distribution

Specifically, let there be a unit mass of heterogeneous individuals, indexed by $j \in [0, 1]$ each endowed with $l\bar{H}$ units of low-skill labor and $\Gamma(j) = \bar{H}\frac{(1+q)}{q}j^{1/q}$ units of high-skill labor (the important assumption here is the existence of a fat tail of individuals with low ability). The parameter $q > 0$ governs the shape of the ability distribution with $q \to \infty$ implying equal distribution of skills and $q < \infty$ implying a ranking of increasing endowments of high-skill on $[0, \bar{H}(1 + q)/q]$. Proposition 3 can be extended to this case and in fact the steady state values $(G^*, \hat{h}^{A*}, g^{N*}, \chi^*)$ are the same as in the model with a fixed high-skill labor supply $\bar{H}$. Proposition 2 also applies except that the asymptotic growth rate of low-skill wages is higher (see Appendix 8.7):

$$g^v_\infty = g^Y_\infty = \psi g^N_\infty \text{ and } g^w_\infty = \frac{1 + q}{1 + q + \beta(\sigma - 1)} g^Y_\infty. \qquad (26)$$

At all points in time there exists an indifferent worker $(\bar{j}_t)$ where $w_t = (1+q)/q(\bar{j}_t)^{1/q}v_t$, with all $j \leq \bar{j}_t$ working as low-skill workers and all $j > \bar{j}_t$ working as high-skill workers. This introduces an endogenous supply response as the diverging wages for low- and high-skill workers encourage shifts from low-skill to high-skill jobs, which then dampens the relative decline in low-skill wages. Hence, besides securing themselves a higher future wage growth, low-skill workers who switch to a high-skill occupation also benefit the remaining low-skill workers. Since all changes in the stock of labor are driven by demand-side effects, wages and employment move in the same direction.

4.2 Bayesian estimation

Because of data availability and to make our exercise easily comparable to the rest of the literature, we focus on the last 50 years. In particular, this allows us to identify low-skill workers with non-college educated workers and high-skill workers with college educated workers. We match the skill-premium and the ratio of skilled to non-skilled workers (both calculated using the methodology of Acemoglu and Autor, 2011) as well as the growth rate of real GDP/employment and the share of labor in total GDP (both taken from the National Income and Products Accounts). We further associate the use of machines with private equipment (real private non-residential equipment, "Table 2.2. Chain-type Quantity Indexes" from NIPA). All time series start in 1963 when the skill-premium and skill-ratio are first available and until 2007 to avoid the Great Recession. We match the accumulated growth rate of private equipment by indexing both X and real private equipment to 100 in 1963.[32] We stress that our exercise is much more demanding than previous attempts which feed in input time paths from the data, while we make them endogenous. Both Katz and Murphy (1992) or Golden and Katz (2008) take as given the time paths of labor inputs (endogenous here) and do not attempt to explain the skill bias of technical change which here is constrained to result from economic incentives and decisions. Similarly, Krusell et al. (2000) do not allow for technological change but take the time paths of labor inputs and equipment as given.

Due to the relatively small sample size we use Bayesian techniques to estimate our model, though little would change if we instead employed Maximum Likelihood procedures (in fact since we choose a uniform prior the maximum likelihood point estimate is equal to the mode of the Bayesian estimator). The model presented until now is not inherently stochastic, and in order to bring it to the data, we add normally distributed auto-correlated measurement errors. That is, we consider an economy where the underlying structure is described deterministically by our model, but the econometrician only observes variables with normally distributed auto-correlated measurement errors. With a full parametrization of the model the parameters are not uniquely identified and we restrict $\bar{H} = 1$ without loss of generality. Therefore, our deterministic model has 14 parameters including n_0 and G_0. Including two parameters (variance and correlation) for each of the five measurement errors, this leaves us with 24 parameters.[33] This gives

[32] The use of machines, X, has no natural units and we can therefore not match the level of X. Alternatively, we could normalize X by GDP, but we do no think of equipment as the direct empirical counter-part of X. First, equipment is a stock, whereas X is better thought of as a flow variable. Second many aspects of automation might not be directly captured in equipment. Hence, equipment is better thought of as a proxy for X that grows in proportion to X. Empirically, equipment/GDP is about twice that of our predicted value of X/GDP.

[33] More specifically, for time period $t = 1, ...T$, let $(Y_1^t, ..., Y_M^t) \in R^{M \times T}$ be a vector of M predicted variables with time paths of $Y_m^t = (Y_{m,s})_{s=1}^t$ for $m \in \{1, ..., M\}$ and $Y^t = (Y_1^t, ..., Y_M^t)$. Let the complete set of parameters in the deterministic model be $b_P \in \mathcal{B}_P \subset R^K$. We can then write the predicted values as $Y_m^T(b_P)$, for $m = 1, ...M$. We add normally distributed measurement errors with zero mean to get the predicted values as $\hat{Y}_m^T = Y_m^T + \epsilon_m^T$, where $\epsilon_m^T \sim N(0, \Sigma_m)$ and Σ_m is the covariance matrix of the

a joint distribution of the observed variables given parameters and, with a chosen prior, standard Bayesian methods can be employed to find the posterior distribution (see Appendix 8.10 for a full description of the procedure as well as domain on the prior uniform distribution). The domain of the prior is deliberately kept wide for parameters not easily recovered from other studies such as the characteristics of the automation technology.

Table 2 shows the mode of the posterior distribution. The unconditional posterior distribution of each parameter is shown in Figure 22 in Appendix 8.10, which demonstrates that variance for the posterior unconditional distribution is generally small.

Table 2: The mode of the posterior distribution.

	σ	μ	β	l	γ	$\tilde{\kappa}$	θ	η	κ	ρ	φ	q
Mode	4.17	0.66	0.76	0.91	0.20	0.29	2.09	0.27	0.72	0.058	8.60	0.82
	n_0	G_0	ρ_1	σ_1^2	ρ_2	σ_2^2	ρ_3	σ_3^2	ρ_4	σ_4^2	ρ_5	σ_5^2
Mode	0.49	0.59	0.97	0.01	0.99	0.026	0.96	0.001	0.24	0.0004	0.97	0.051

Note: (σ_i^2, ρ_i), $i = 1, ..., 5$ estimate refer to skill-premium, skill-ratio, labor share of GDP, growth rate of GDP/employment, and Real Private Equipment, respectively.

Three parameter estimates are worth noting. First, the parameter of the automation externality, $\tilde{\kappa}$, is centered around 0.29 implying a substantial automation externality, a force for an accelerated Phase 2. Second, G_0 is centered around 0.59 implying that Phase 2 was already well underway in the early 1960s. Finally, the estimate of β—the factor share to machines/low-skill workers— of 0.76 implies substantial room for automation.

Figure 7 further shows the predicted path of the matched data series along with their empirical counterparts at the mode of the posterior distribution. Panel A demonstrates that the model matches the rise in the skill-premium from the late 1970s onwards reasonably well, but misses the flat skill-premium in the period before. As argued in Goldin and Katz (2008), the flat skill-premium in this period is best understood as the consequence of a large increase in the stock of college-educated workers caused by other factors than technological change (the Vietnam war and the increase in female college enrollment). Correspondingly, our model, which only allows relative supply to respond to relative factor rewards, fails to capture a substantial increase in the relative stock

measurement errors. The errors are independent across types, $E[\epsilon_m^{T'} \epsilon_n^T] = 0$, for $m \neq n$, but potentially auto-correlated: the elements of Σ_m are such that the t, t'-element of Σ_m is given by $\sigma_m^2 \rho_m^{|t-t'|}$, where $\sigma_m^2 > 0$ and $-1 < \rho_m < 1$. Hence, σ_m^2 is the unconditional variance of a measurement error for variable m and ρ_m is its auto-correlation. This gives a total of $2M$ stochastic parameters and we label the combined set of these and b_P as $b \in \mathcal{B} \subset R^{2M+K}$. This leads to a joint probability density for $\hat{Y}^T$ of $f(\hat{Y}^T|b) = \Pi_{m=1}^M f_m(\hat{Y}_m^T|b)$ and with a uniform prior $f(b|\hat{Y}^T) \propto f(\hat{Y}^T|b)$.

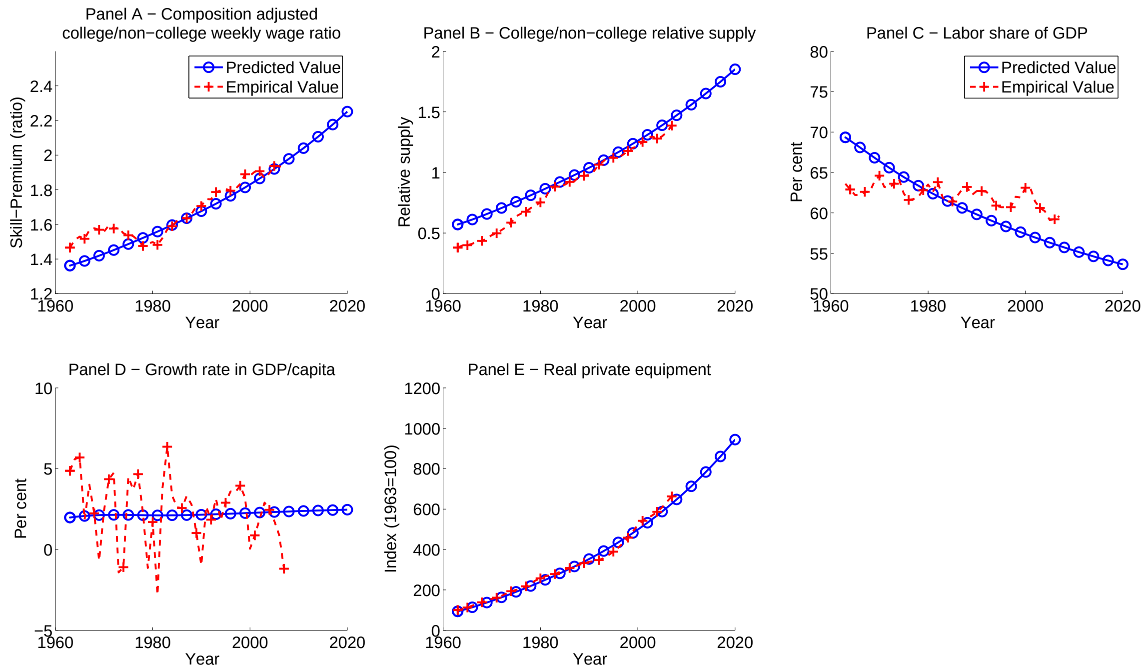

Figure 7: Predicted and Empirical time paths

of skilled labor in the 1960s and early 1970s (Panel B). More importantly, the model predicts a substantially higher drop in the labor share of GDP (14 versus 5 percentage points empirically). The simple structure of the model forces any increase in the use of machines to be reflected in a drop in the labor share. As discussed in Section 3.6, a number of extensions would allow for more flexibility. The model matches the average growth rate of GDP/employment, but as a long-run growth model, is obviously not capable of matching the short-run fluctuations around trend (Panel D).

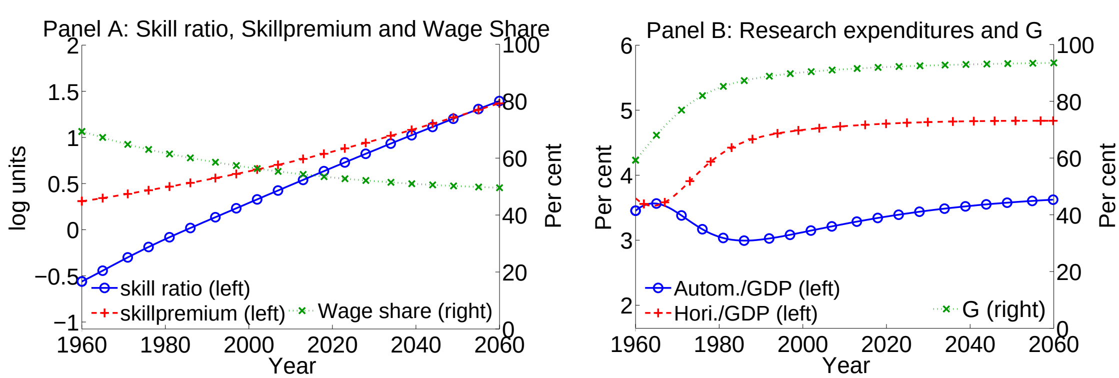

Figure 8: Transitional Dynamics with calibrated parameters

Panel E shows that the model captures the exponential growth in private equipment very well—this is not an automatic consequence of matching the GDP/employment growth rate as equipment has been growing at around 1 percentage point faster than GDP since 1963.

Figure 8 plots the transitional dynamics from 1960 to 2060. Panel A shows that the skill ratio and the skill premium are predicted to keep growing at nearly constant rates, while the labor share is to stabilize at a slightly lower level than today. Panel B suggests that the share of automated products today is not far from its steady state value.

5 Middle-Skill Workers and Wage Polarization

As mentioned in the introduction, a recent literature (e.g. Autor et. al., 2006 and Autor and Dorn, 2013) argues that since the 1990s, wage polarization has taken place: inequality has kept rising in the top half of the distribution, but it has narrowed for the lower half. They conjecture that these "middle-skill"-workers are performing cognitive routine tasks which are the most easily automated. Our model suggests a related, but distinct explanation: automating the tasks performed by middle-skill workers is not easier, but more difficult and therefore happened later (or alternatively, the easily automatable low-skill tasks have already been automated). Hence, before 1990 and in fact for most of the 20^{th} century low-skill workers were in the process of being replaced by machines as semi-automated factories, mechanical farming, household appliances etc were increasingly used, whereas since the 1990s, computers are replacing middle-skill workers. In fact, Figure 3 in Autor and Dorn (2013) shows that low-skill workers left non-service occupations from the 70's, which is consistent with the view that their tasks in non-service occupations were automated before the middle-skill workers' tasks. As such our model can explain how a phase of wage polarization can follow one of a uniform increase in wage inequality.

To make this precise, we introduce a mass M of middle-skill workers into the model. These workers are sequentially ranked such that high-skill workers can perform all tasks, middle-skill workers can perform middle-skill and low-skill tasks, and low-skill workers can perform only low-skill tasks. All newly introduced intermediate products continue to be non-automated, but there is an exogenous probability δ that they require low-skill and high-skill tasks as described before, and a probability $1-\delta$ that they require both middle-skill and high-skill tasks in an analogous manner. We refer to the former type of products as "low-skill products" and the latter type as "middle-skill products". This gives the following production functions (for $i \in [0, N_t]$):

$$y_L(i) = \left[l(i)^{\frac{\epsilon-1}{\epsilon}} + \alpha(i) \left(\tilde{\varphi}_L x(i) \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon\beta}{\epsilon-1}} h(i)^{1-\beta},$$

$$y_M(i) = \left[m(i)^{\frac{\epsilon-1}{\epsilon}} + \alpha(i) \left(\tilde{\varphi}_M x(i) \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon\beta}{\epsilon-1}} h(i)^{1-\beta},$$

where $y_L(i)$ and $y_M(i)$ are the production of low-skill and middle-skill products, respectively, and $m(i)$ is the use of middle-skill workers by a firm of the latter type. $\tilde{\varphi}_L$ and $\tilde{\varphi}_M$ are the productivity of machines that replace low-skill and middle-skill workers, respectively. The mass of low-skill products is δN, the mass of middle-skill products is $(1 - \delta)N$ (alternatively all products could be produced by all factors; this would make the analysis substantially more complicated without altering the underlying argument). The final good is still produced competitively by a CES aggregator of all intermediate inputs, and all machines are produced one-for-one with the final good keeping a constant price of 1. The shares of automated products, G_L and G_M will in general differ.

Both types of producers have access to an automation technology as before, but we allow the productivity to differ, such that automation happens with intensity $\eta_L G_L^{\widetilde{\kappa}} (N h_L^A)^\kappa$ for low-skill products and $\eta_M G_M^{\widetilde{\kappa}} (N h_M^A)^\kappa$ for middle-skill products. The equilibrium is defined analogously to section 3.3 and a proposition analogous to Proposition 3 exists.

We choose $\delta = 1/2$ and set $L = M = 1/3$ and keep parameters as before except that we choose $\widetilde{\varphi}_M = 0.15$ and $\tilde{\varphi}_L = 0.3$, to focus on a situation where machines are less productive in middle-skill products than in low-skill ones. The situation would be similar had we chosen $\widetilde{\varphi}_M = \tilde{\varphi}_L$, but $\eta_M < \eta_L$ such that the automation technology for middle-skill firms is less productive. Figure 9 describes the equilibrium in the presence of a large externality in the automation technology ($\widetilde{\kappa} = 0.5$).

The overall picture is similar to that of Figure 4, but with distinct paths for low-skill and middle-skill wages denoted w and u. One can now distinguish 4 phases. Phase 1 is analogous to Phase 1 in the previous case, and all wages grow at roughly the same rate. From around year 200, low-skill wages become sufficiently high, that low-skill product firms start investing in automation and G_L starts growing. Yet, since machines are less productive in middle-skill workers' tasks, G_M stays low until around year 300. During this second phase, inequality increases uniformly, high-skill wages grow faster than middle-skill wages which again grow faster than low-skill wages. Middle-skill wages do not grow as fast as GDP because automation in low-skill products increases their market share at the expense of the middle-skill products. From around year 300, the economy enters a third phase, where automation in middle-skill products is now intense. As a result, the growth rate of middle-skill wages drops further, such that low-skill wages actually grow faster than middle-skill wages (all along $v_t \geq u_t \geq w_t$, so no group has an incentive to be employed below its skill level).[34] However, depending on parameters, the polarization phase need not be as salient as here (for instance, there is barely any polarization when there is no externality in automation, $\tilde{\kappa} = 0$, but the other parameters are kept identical, see Appendix 7.2.5 for this case).

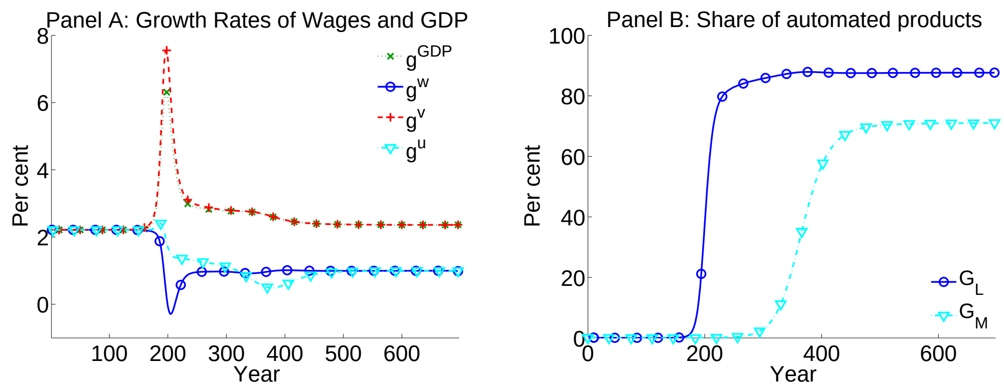

Figure 9: Transitional dynamics with middle-skill workers in the presence of an automation externality ($\tilde{\kappa} = 0.5$).

Finally, in a fourth Phase (from around year 450), G_L and G_M are close to their steady state levels and the economy approaches the asymptotic steady state, with low-skill and middle-skill wages growing positively but at a rate lower than that of the economy. Proposition 2 can be extended to this case. High-skill wages and output all grow at the same rate which depends on the growth rate of the number of products, while low-skill and middle-skill wages grow at a lower rate such that:

$$g_{\infty}^{v} = g_{\infty}^{Y} = \psi g_{\infty}^{N} \text{ and } g_{\infty}^{w} = g_{\infty}^{u} = g_{\infty}^{Y} / \left(1 + \beta(\sigma - 1)\right).$$

Our model shows how automation may affect more middle-skill workers than low-skill workers, because the economic benefits of automating the former are greater despite a worse automation technology. Yet, some papers argue that the technological opportunities for automation themselves are today lower for low-skill than for middle-skill workers.

[34]Empirically, the polarization looks more like a J curve than a U curve as the difference in growth rates of wages between the bottom and the middle of the income distribution is modest. Here as well, high-skill wages grow faster than both low-skill and middle-skill wages from the beginning of Phase 2.

This is easy to reconcile with our model: assume that for both types of products, a common fixed share can never be automated (for instance because the associated tasks are not routine enough). After the start of Phase 2, the share of low-skill workers hired in products that can never be automated will be larger than the corresponding share for middle-skill workers (since a higher share of low-skill products will already have been automated), so that it will be on average easier to automate a middle-skill product than a low-skill one. That is, middle-skill workers end up performing on average more routine tasks that are currently being automated (as emphasized by the literature), but this is only the case *because* the routine tasks that were performed by low-skill workers and that could be automated have already been largely automated.

Naturally, the phase of intense automation of middle-skill products may occur sooner than that of low-skill products: for instance if the supply of middle-skill workers is low enough to generate a large middle-skill over low-skill wage ratio. In fact, Katz and Margo (2014) suggest that the recent phase of polarization has a counterpart in the 19^{th} century de-skilling of manufacturing as the tasks of (middle-skilled) artisans got automated, as their wages were much higher than that of unskilled workers (maybe because urbanization increased the supply of low-skill workers).

Automating some high-skill tasks. Alternatively, one may assume that middle-skill and high-skill workers are identical and therefore use this framework to analyze the case where some high-skill production tasks are automatable—in particular, if β is close to (but smaller than) 1, such a model would capture the situation where all production tasks except for headquarter services are automatable. Then, automation will affect in turn both low-skill and high-skill products, with the order depending on the relative wage of both and the relative effectiveness of the two automation technologies. One crucial difference with the middle-skill case is that as some high-skill workers' tasks remain non-automatable (in production and in research), their wage in the long-run still grows at the same rate as GDP so that the skill premium keeps rising.

6 Conclusion

In this paper, we introduced automation in a horizontal innovation growth model. We showed that in such a framework, the economy will undertake a structural break. After an initial phase with stable income inequality and stable factor shares, automation picks up. During this second phase, the skill premium increases, low-skill wages stagnate and possibly decline, the labor share drops—all consistent with the US experience in the past 50 years—and growth starts relying increasingly on automation. In a third phase, the share of automated products stabilizes, but the economy still features a constant shift of low-skill employment from recently automated firms to as of yet non-automated firms. With a constant and finite aggregate elasticity of substitution between low-skill workers and machines, low-skill wages grow in the long-run. Wage polarization can be accounted for once the model is extended to include middle-skill workers. In addition we saw that an intense phase of automation need not be associated with an increase in economic growth and that a more productive automation technology increases low-skill wages in the long-run.

The model shows that there is a long-run tendency for technical progress to displace substitutable labor (this is a point made by Ray, 2014, in a critique of Piketty, 2014), but this only occurs if the wages of the workers which can be substituted for are large relative to the price of machines. This in turn can only happen under three scenarios: either automation must itself increase the wages of these workers (the scale effect dominates the substitution effect), or there is another source of technological progress (here, horizontal innovation), or technological progress allows a reduction in the price of machines relative to the consumption good (here, only present in Appendix 8.8). Importantly, when machines are produced with a technology similar to the consumption good, automation can only reduce wages temporarily: a prolonged drop in wages would end the incentives to automate in the first place.

Fundamentally, the economy in our model undertakes an endogenous structural change when low-skill wages become sufficiently high. This distinguishes our paper from most of the literature, which seeks to explain changes in the distribution of income inequality through exogenous changes: an exogenous increase in the stock of equipment as per Krusell et al. (2000), a change in the relative supply of skills, as per Acemoglu (1998), or the arrival of a general purpose technology as in the associated literature. This makes our paper closer in spirit to the work of Buera and Kaboski (2012), who argue that the increase in income inequality is linked to the increase in the demand for high-skill intensive services, which results from non-homotheticity in consumption.

The present paper is only a first step towards a better understanding of the links between automation, growth and income inequality. In future research, we will extend it to consider policy implications. The simple sensitivity analysis on the automation technology (section 3.7) suggests that capital taxation will have non-trivial implications in this context. Automation and technological development are also intrinsically linked to the international economy. Our framework could be used to study the recent phenomenon of "reshoring", where US companies that had offshored their low-skill intensive activities to China, now start repatriating their production to the US after having further automated their production process. Finally, our framework could also be used to study the impact of automation along the business cycle: Jaimovich and Siu (2012) argue that the destruction of the "routine" jobs happens during recessions, which raises the question of whether automation is responsible for the recent "jobless recovery".

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7 Main Appendix (For Online Publication)

7.1 Formal description of the normalized system of differential equations

Here, we derive the system of differential equations satisfied by the normalized variables (n_t, G_t, h_t, χ_t). The definition of n_t immediately gives:

$$\dot{n}_t = -\frac{\beta}{(1-\beta)(1+\beta(\sigma-1))} g_t^N n_t. \tag{27}$$

Rewriting (20) with $\hat{h}_t^A$ gives:

$$\dot{G}_t = \eta G_t^{\widetilde{\kappa}} \left(\hat{h}_t^A\right)^{\kappa} (1-G_t) - G_t g_t^N. \tag{28}$$

Then, plug (21) (with equality) and (19) into (18) and use the definition of $\widehat{h}_t^A$ to get

$$r_t v_t = \gamma N_t \pi_t^N + \gamma \frac{1-\kappa}{\kappa} v_t \widehat{h}_t^A + \dot{v}_t - g_t^N v_t. \tag{29}$$

Next, take the difference between (17) and (18) to obtain

$$r_t \left(V_t^A - V_t^N\right) = \left(1 - \omega_t \left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{-\mu}\right) \pi_t^A + v_t h_t^A - \eta G_t^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa} \left(V_t^A - V_t^N\right) + \left(\dot{V}_t^A - \dot{V}_t^N\right),$$

using (19) and the definition of $\widehat{h}_t^A$ allows to rewrite this equation as:

$$r_t v_t = \begin{array}{c} \eta\kappa G_t^{\widetilde{\kappa}} N_t \left(\widehat{h}_t^A\right)^{\kappa-1} \left(\left(1 - \omega_t \left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{-\mu}\right) \pi_t^A - \frac{1-\kappa}{\kappa} \frac{v_t}{N_t} \widehat{h}_t^A\right) \\ + \dot{v}_t - v_t \widetilde{\kappa} \frac{\dot{G}_t}{G_t} - v_t g_t^N + v_t (1-\kappa) \frac{\dot{\widehat{h}}_t^A}{\widehat{h}_t^A} \end{array}.$$

Using (28) and (29) gives:

$$\gamma N_t \omega_t \left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{-\mu} \pi_t^A + \gamma \frac{1-\kappa}{\kappa} v_t \widehat{h}_t^A = \begin{array}{c} \eta\kappa G_t^{\widetilde{\kappa}} N_t \left(\widehat{h}_t^A\right)^{\kappa-1} \\ \times \left(\left(1 - \omega_t \left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu}\right) \pi_t^A - \frac{1-\kappa}{\kappa} \frac{v_t}{N_t} \widehat{h}_t^A\right) \\ - v_t \widetilde{\kappa} \left(\eta G^{\widetilde{\kappa}-1} \left(\widehat{h}_t^A\right)^{\kappa} (1-G_t) - g_t^N\right) + v_t (1-\kappa) \frac{\dot{\widehat{h}}_t^A}{\widehat{h}_t^A} \end{array}.$$

Rearranging terms, using the definition of $\widehat{v}_t$ and defining normalized profits $\widehat{\pi}_t^A \equiv N_t^{1-\psi}\pi_t^A$ allows us to rewrite this equation as

$$\dot{\hat{h}}_t^A = \begin{array}{c} \gamma\frac{\widehat{h}_t^A}{1-\kappa}\left(\omega_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{-\mu}\frac{\widehat{\pi}_t^A}{\hat{v}_t}+\frac{1-\kappa}{\kappa}\widehat{h}_t^A\right) \\ -\frac{\eta\kappa G_t^{\widetilde{\kappa}}\left(\widehat{h}_t^A\right)^{\kappa}}{1-\kappa}\left(1-\omega_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{-\mu}\right)\frac{\hat{\pi}_t^A}{\hat{v}_t}, \end{array} \tag{30}$$

$$+\,\eta G_t^{\widetilde{\kappa}}\left(\widehat{h}_t^A\right)^{\kappa+1}+\frac{\widetilde{\kappa}}{1-\kappa}\left(\eta G_t^{\widetilde{\kappa}-1}\left(\widehat{h}_t^A\right)^{\kappa+1}(1-G_t)-g_t^N\hat{h}_t^A\right)$$

Rewriting (22) using the definition of $\widehat{c}_t$, leads to

$$r_t = \rho + \theta\frac{\dot{\widehat{c}_t}}{\widehat{c}_t} + \theta\psi g_t^N.$$

Combining this equation with (29), and using the definitions of $\chi_t, \widehat{v}_t$ and $\widehat{\pi}_t^A$ leads to

$$\dot{\chi}_t = \chi_t\left(\gamma\omega_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{-\mu}\frac{\widehat{\pi}_t^A}{\hat{v}_t}+\gamma\frac{1-\kappa}{\kappa}\widehat{h}_t^A-\rho-(\theta\psi-\psi+1)\,g_t^N\right). \tag{31}$$

Together equations (27), (28), (30) and (31) form a system of differential equations which depends on ω_t, $\widehat{\pi}_t^A/\widehat{v}_t$ and g_t^N. To determine $\widehat{\pi}_t^A/\widehat{v}_t$, recall that (as proved in the text), profits are given by

$$\pi\left(w,v,\alpha\left(i\right)\right) = \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}c\left(w,v,\alpha\left(i\right)\right)^{1-\sigma}Y.$$

Using (4) and the definition of ω_t, one gets:

$$\pi_t^A = \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}\left(\beta^\beta(1-\beta)^{1-\beta}\right)^{\sigma-1}\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}v_t^{-\psi^{-1}}Y_t. \tag{32}$$

Rearranging terms in (11) gives

$$\widehat{v}_t = \left(\frac{\sigma-1}{\sigma}\right)^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}(1-\beta)\left(G\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}+(1-G)\,\omega_t\right)^{\psi}. \tag{33}$$

Using (8), one further gets:

$$Y_t = \sigma\psi\widehat{v}_t H_t^P N_t^{\psi}. \tag{34}$$

Therefore, rewriting (32) with (33) and (34), one gets:

$$\frac{\widehat{\pi}_t^A}{\widehat{v}_t} = \frac{\psi\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} H_t^P}{G\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} + (1-G)\,\omega_t}, \tag{35}$$

which still requires finding H_t^P. Using (4), (5), (6) and aggregating over all automated firms, one gets the following expression for the total demand of machines:

$$X_t = \beta G_t N_t \varphi \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \left(\beta^{\beta}\,(1-\beta)^{(1-\beta)}\right)^{\sigma-1} \left(\omega^{\frac{1}{\mu}} + \varphi\right)^{\mu-1} v_t^{-\psi^{-1}} Y_t.$$

Using (33), this expression can be rewritten as:

$$X_t = \beta G_t \frac{\sigma-1}{\sigma} \frac{\varphi\left(\omega_t^{\frac{1}{\mu}} + \varphi\right)^{\mu-1}}{G\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} + (1-G)\,\omega_t} Y_t. \tag{36}$$

This together with (34) implies that $\widehat{c}_t$ obeys

$$\widehat{c}_t = \left(1 - \beta\frac{\sigma-1}{\sigma} \frac{G_t\varphi\left(\omega_t^{\frac{1}{\mu}} + \varphi\right)^{\mu-1}}{G_t\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} + (1-G_t)\,\omega_t}\right) \sigma\psi\widehat{v}_t H_t^P.$$

Combining this equation with the definition of χ_t and (33), leads to

$$H_t^P = \left(\frac{\sigma-1}{\sigma}\right)^{\frac{1}{1-\beta}\left(\frac{1}{\theta}-\beta\right)} \times \frac{(1-\beta)^{\frac{1}{\theta}}\beta^{\frac{\beta}{1-\beta}\left(\frac{1}{\theta}-1\right)}\chi_t^{\frac{1}{\theta}}\left(G_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}+(1-G_t)\omega_t\right)^{\psi\left(\frac{1}{\theta}-1\right)+1}}{G_t\left(\left(1-\beta\frac{\sigma-1}{\sigma}\right)\varphi+\omega_t^{\frac{1}{\mu}}\right)\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu-1}+(1-G_t)\omega_t}. \tag{37}$$

Using the definition of H_t^D and $\widehat{h}_t^A$, one can rewrite (16) for high-skill workers as:

$$g_t^N = \gamma\left(H - H_t^P - (1-G_t)\,\widehat{h}_t^A\right). \tag{38}$$

Together (35), (37) and (38) determine $\widehat{\pi}_t^A/\widehat{v}_t$ and g_t^N as a function of the original variables $n_t, G_t, \widehat{h}_t^A, \chi_t$ and of ω_t, which still needs to be determined. To do so, combine (10) and (11), and use the definitions of n_t and ω_t to obtain an implicit definition of ω_t:

$$\omega_t = n_t \left[\begin{array}{c} \left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}} \frac{H_t^P}{L} \left(G_t \left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu-1} \omega_t^{\frac{1-\mu}{\mu}} + (1-G_t) \right) \\ \times \left(G_t \left(\omega_t^{\frac{1}{\mu}} + \varphi\right)^{\mu} + (1-G_t)\omega_t \right)^{\psi-1} \end{array} \right]^{\frac{\beta(1-\sigma)}{1+\beta(\sigma-1)}}. \tag{39}$$

Therefore eventually the system of differential equations satisfied by $n_t, G_t, \widehat{h}_t^A, \chi_t$ is defined by (27), (28), (30) and (31), with $\widehat{\pi}_t^A/\widehat{v}_t$, H_t^P, g_t^N and ω_t given by (35), (37), (38) and (39).

7.2 Complements on simulation

7.2.1 Wealth and consumption

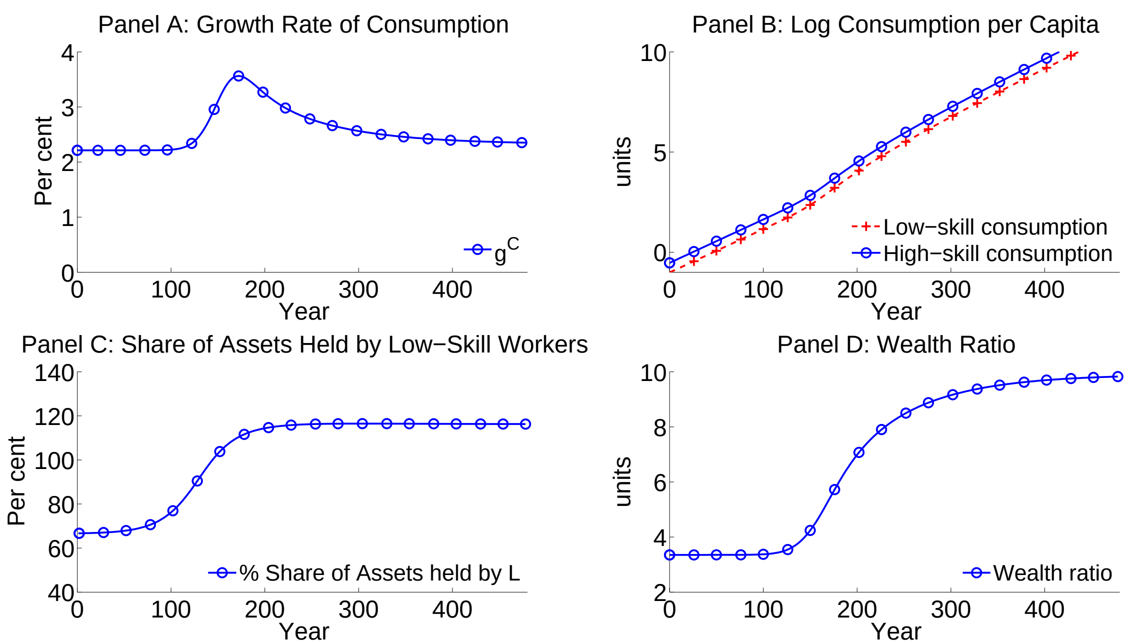

Figure 10: Consumption and wealth for baseline parameters. Panel A shows yearly growth rates for consumption, Panel B log consumption of high-skill workers and low-skill workers (per capita), Panel C the share of assets held by low-skill workers and Panel D the wealth to GDP ratio.

Figure 10 shows the evolution of wealth and consumption for the baseline parameters both in the aggregate and for each skill group. Panel A shows that consumption growth follows a pattern very similar to that of GDP growth (displayed in Figure 2.A), which is in line with a stable ratio of total R&D expenses over GDP across the three phases (Figure 2.D). In the absence of any financial constraints, low-skill and high-skill consumption must grow at the same rate, with high-skill workers consuming more since they have a higher income (Panel B). Since low-skill labor income becomes a negligible share of GDP, while the high-skill labor share increases, a constant consumption ratio can only be achieved if high-skill workers borrow from low-skill workers in the long-run. This is illustrated in Panel C, which shows the share of assets held by low-skill workers, under the assumption that initially assets holdings per capita are identical for low-skill and high-skill workers (so that low-skill workers hold 2/3 of the assets in year 0, since with these parameters $H/L = 1/2$). Initially, low-skill and high-skill income grow at a constant rate so that the share of assets held by low-skill workers is stable; but, in anticipation of a lower growth rate for low-skill wages than for high-skill wages, low-skill workers start saving more and more, and the share of assets they hold increases. This share eventually reaches more than 100%, meaning that the high-skill workers net worth becomes negative. As claimed in the text, Panel D shows that since profits become a higher share of GDP (an effect which dominates a temporary increase in the interest rate in Phase 2), the wealth to GDP ratio increases in phase 2, such that its steady state value is nearly 3 times higher than its original value.

The accumulation of asset holdings by low-skill workers predicted by the model seems counter-factual, it results from our assumptions of infinitely lived agents with identical discount rates and no financial constraints. Reversing these unrealistic assumptions would change the evolution of the consumption side of the economy but should not alter the main results which are about the production side.

7.2.2 A delayed decline in the labor share

Empirically, the drop in the labor share is a more recent phenomenon than the increase in the skill premium. In Figure 11, we choose parameters such that this happens. The automation technology is more productive $\eta = 0.4$; the automation technology is less concave $\kappa = 0.9$ (so that a higher incentive to automate is required to get a significant share of high-skill workers in automation innovation); and all other parameters are identical to the baseline case. In this case, more high-skill workers get allocated to automation during Phase 2: as shown in Panel C automation expenditures represent a much larger share of GDP during Phase 2 than they do in the baseline case. The mass of high-skill workers engaged in production declines during Phase 2. This results first in a sharper increase in the skill premium (the skill premium condition moves further to the left). In addition, the drop in the labor share is delayed since innovation spending are part of GDP while capital income is a constant share of output Y. The growth rates of low-skill and high-skill wages start diverging significantly from around year 135 and by year 150, the high-skill wage growth rate is 2pp higher than the low-skill wage growth rate, while the total labor share only start declining from around year 150 and in fact increases slightly before.

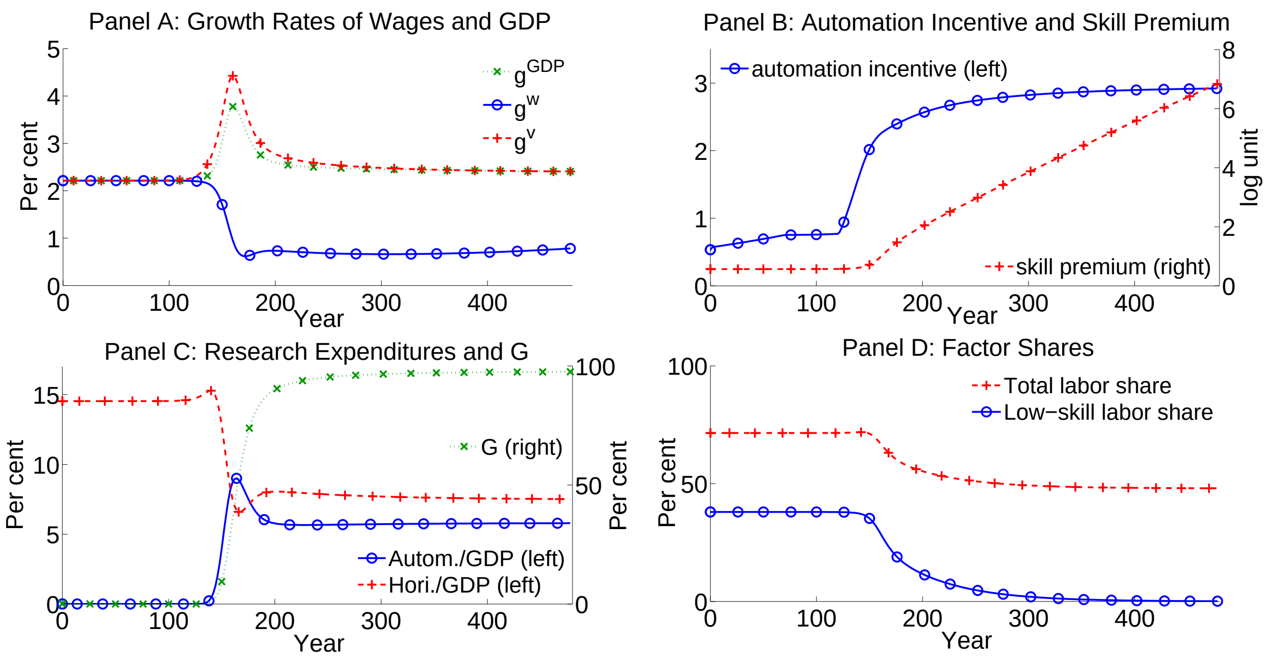

Figure 11: Transitional dynamics with a delayed drop in capital share

7.2.3 Negative growth for low-skill wages without the automation externality

Figure 12 shows the transitional dynamics for a case without the automation externality but where low-skill wages slightly decline for a short time period (our numerical investigation suggests that larger declines in the absence of an automation externality need to be associated with periods where horizontal innovation completely ceases). The associated parameters are given in Table 3.

Table 3: Baseline Parameter Specification

σ	ϵ	β	H	L	θ	η	κ	$\tilde{\varphi}$	ρ	$\widetilde{\kappa}$	γ
3	73	0.72	0.35	0.65	2	0.2	0.97	0.25	0.022	0	0.28

The crucial parameter change is an increase in κ, such that the automation technology is less concave. This delays Phase 2, which is then more intense and leads to a sharp increase in high-skill wages, reducing considerably horizontal innovation (note that in the period where low-skill wages decline the share of high-skill workers hired in production increases slightly, which has a positive contribution on low-skill wages' growth rates).

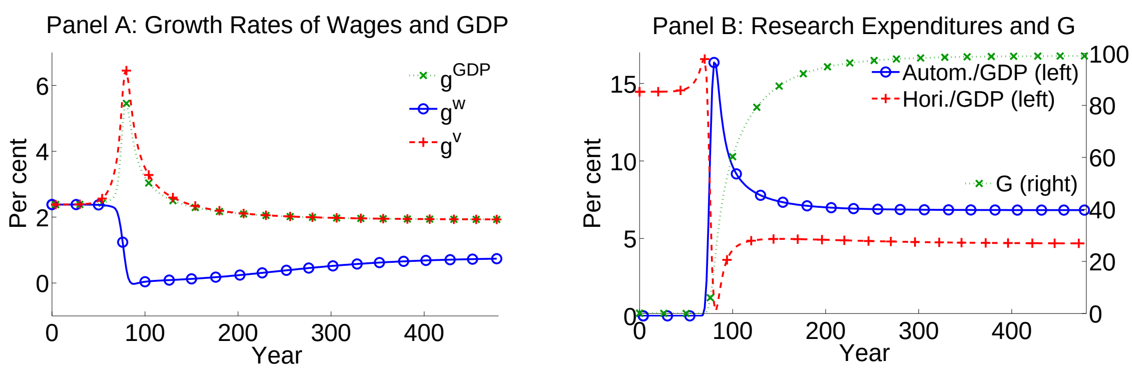

Figure 12: Transitional dynamics with temporarily declining low-skill wages without an automation externality.

7.2.4 Systematic comparative statics

In this section we carry a systematic comparative exercise with respect to the parameters of the model, namely $\sigma, \epsilon, \beta, \rho, \theta, \widetilde{\varphi}, \eta, \kappa, \tilde{\kappa}, \gamma, H/L$ (we keep $H + L = 1$), N_0, G_0. We show the evolution of the growth rate of high-skill and low-skill wages and the share of automated products for the baseline parameters and two other values for one parameter, keeping all the other ones fixed. In all cases, the broad structure of the transitional dynamics in three phases is maintained.

Figures 13.A,B,C show that a higher elasticity of substitution across products σ reduces the growth rate of the economy (the elasticity of output with respect to the number of products is lower), which leads to a delayed transition. The asymptotic growth rate of low-skill wages is a smaller fraction of that of high-skill wages (following Proposition 2), since automated products are a better substitute for non-automated ones. Figures 13.D,E,F show that the elasticity of substitution between machines and low-skill workers in automated firms, ϵ, plays a limited role (as long as the assumption

$\mu < 1$ is kept), a higher elasticity reduces the growth of low-skill wages and increases that of high-skill wages during Phase 2. Figures 13.G,H,I show that a lower factor share in production for high-skill workers (a higherβ) increases the growth rate of the economy (high-skill wages are lower which favors innovation). As a result Phase 2 occurs sooner. In addition, following Proposition 2, the asymptotic growth rate of low-skill wages is a lower fraction of that of high-skill wages (the cost advantage of automated firms being larger).

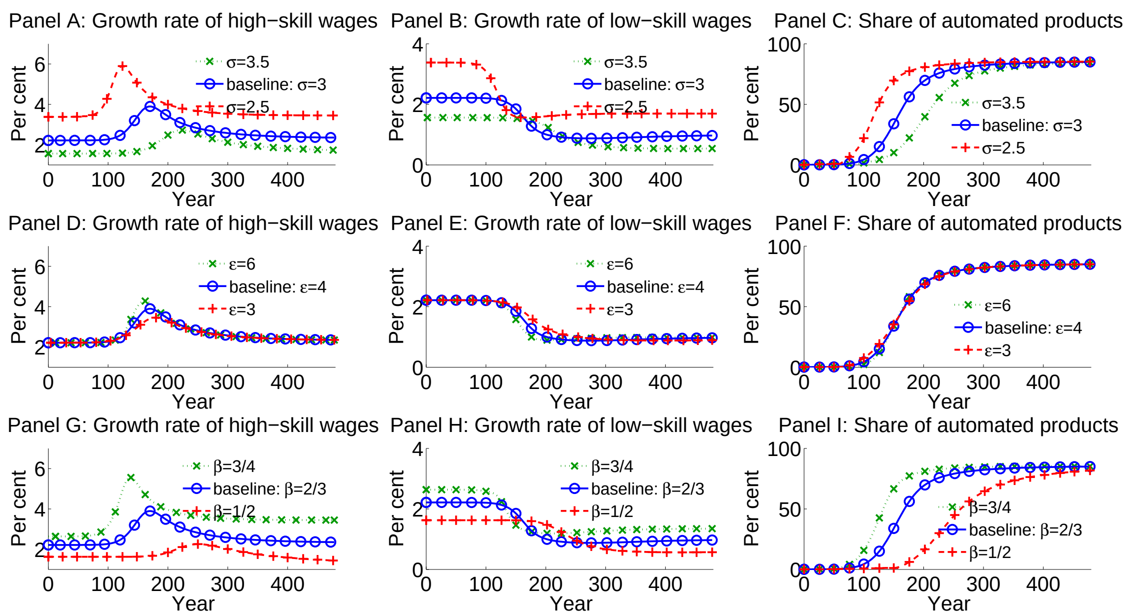

Figure 13: Comparative statics with respect to the elasticity of substitution across products (σ), the elasticity of substitution between machines and low-skill workers in automated firms (ϵ) and the factor share of low-skill workers and machines in production (β).

Figures 14.A,B,C show that a higher discount rate ρ reduces the growth rate of the economy, which slightly postpones Phase 2. At the time of Phase 2, the growth rate of low-skill wages is not affected much by the discount rate: on one hand, since low-skill wages are lower Phase 2 is postponed, which favor low-skill wages' growth, but on the other hand, horizontal innovation is lower which negatively affects low-skill wages. A lower elasticity of intertemporal substitution (a higher θ) has a similar effect on the economy's growth rate (Figures 14.D,E,F). Figures 14.G,H,I show that the productivity of machines ($\widetilde{\varphi}$) only affects the timing of Phase 2 (Phase 2 occurs sooner when machines are more productive).

The comparative statics with respect to the automation technology shown in Figures 15.A,B,C follow the pattern described in the text. A less concave automation technology (higherκ) delays Phase 2 and reduces the economy's growth rate.

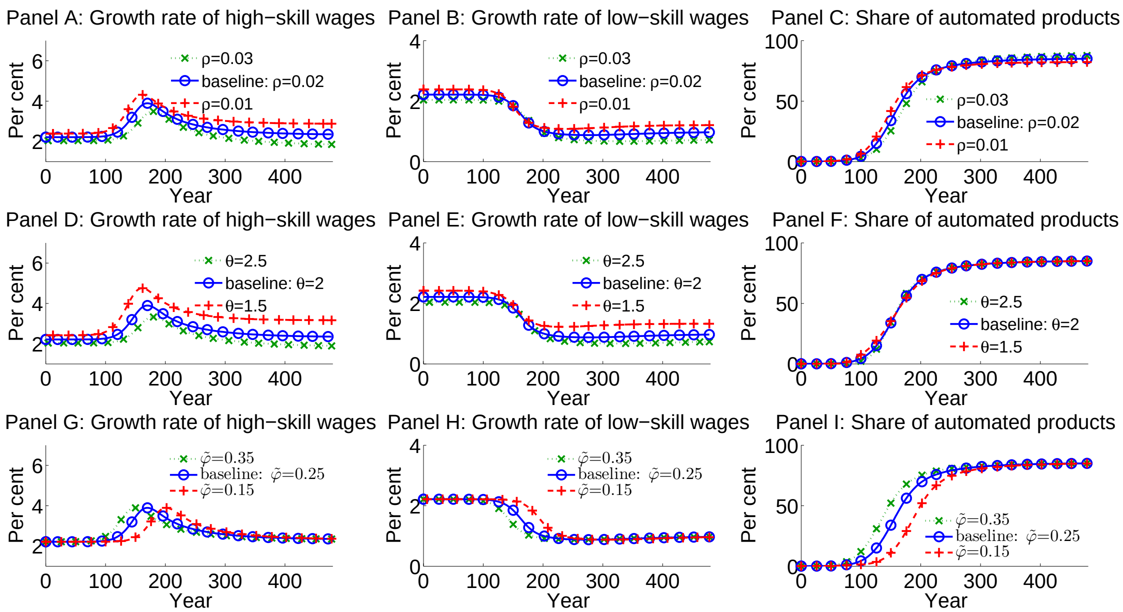

Figure 14: Comparative statics with respect to the discount rate (ρ), the inverse elasticity of intertemporal substitution (θ) and the productivity of machines ($\tilde{\varphi}$)

It particularly affects the growth rate of low-skill wages in Phase 2 (as the increase in automation expenses comes more at the expense of horizontal innovation)—see Figures 15.D,E,F. The role of the automation externality has already been discussed in the text, Figures 15.G,H,I reveal that for a mid-level of the automation externality ($\tilde{\kappa} = 0.25$), the economy looks closer to the economy without the automation externality than to the economy with a large automation externality.

Figures 16.A,B,C show the impact of the horizontal innovation parameter γ, which was already discussed in the text. Figures 16.D,E,F show that a higher ratio H/L naturally leads to a higher growth rate, which implies that Phase 2 occurs sooner. Figures 17.A,B,C show that a higher initial number of products simply advance the entire evolution of the economy. Figures 17.D,E,F show that a higher initial value for the share of automated products barely affects the evolution of the economy, the share of automated products initially drops quickly as there is little automation to start with.

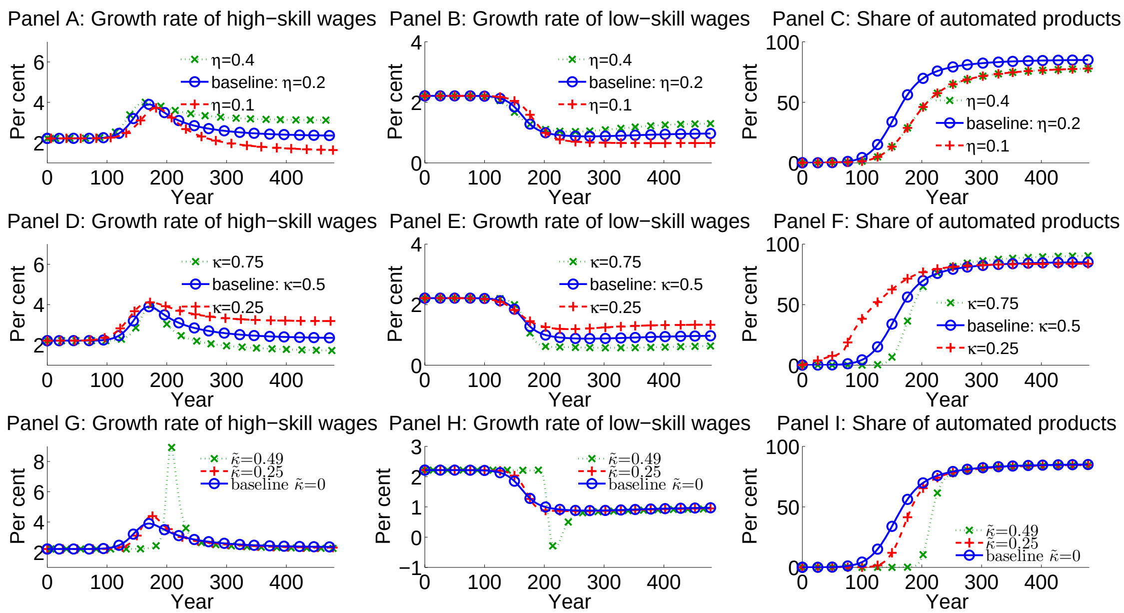

Figure 15: Comparative statics with respect to the automation productivity (η), the concavity of the automation technology (κ) and the automation externality ($\tilde{\kappa}$)

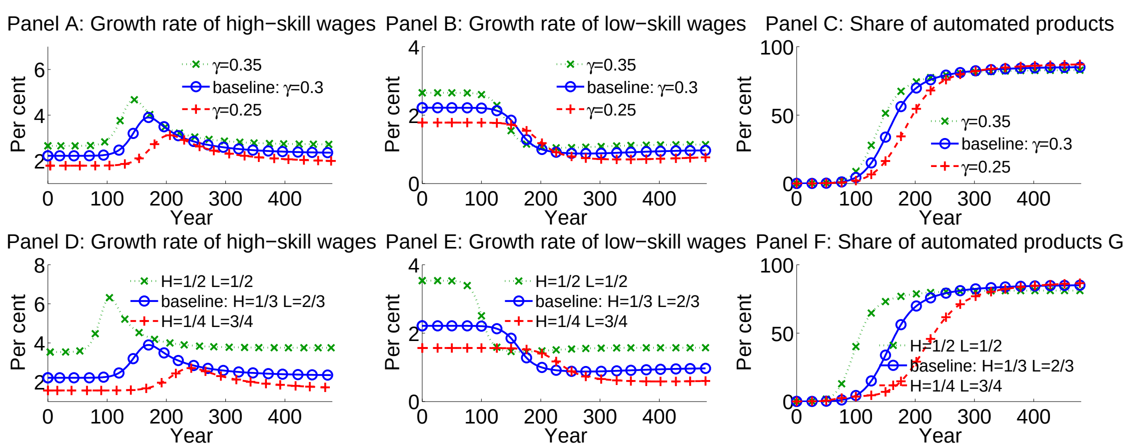

Figure 16: Comparative statics with respect to the horizontal innovation productivity (η) and the skill ration (H/L)

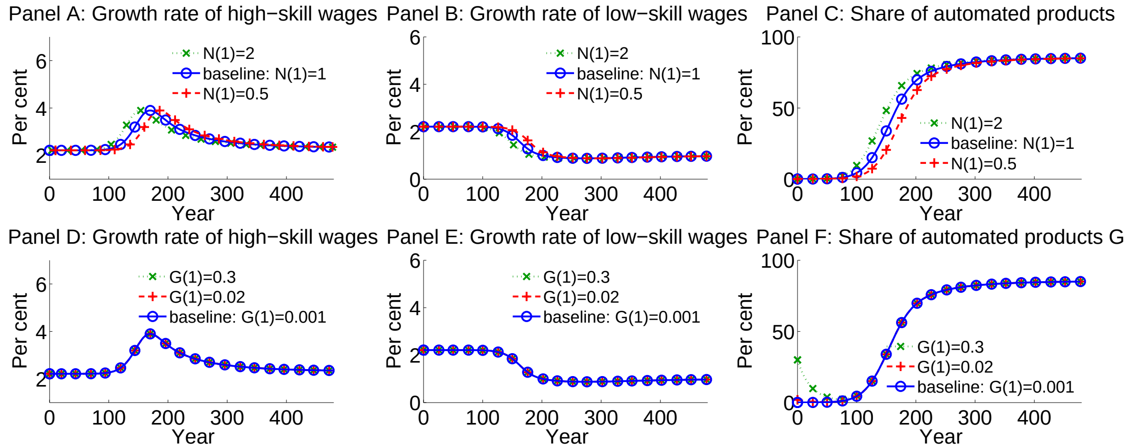

Figure 17: Comparative statics with respect to the initial number of products N_0 and the initial share of automated products G_0.

7.2.5 Case with middle-skill workers but not externality in the automation technology

Figure 18 shows the transitional dynamics in the presence of middle-skill workers for the same parameters as in Figure 9 except that there is no externality in automation ($\tilde{\kappa} = 0$). The two figures are similar. Automation occurs sooner both for low-skill and middle-skill products (as the automation technology is better), and therefore more gradually. As a result, the growth rates of low-skill wages in Phase 2 and middle-skill wages in Phase 3 do not drop as much, such that Phase 3 barely features polarization (g_t^u gets very slightly below g_t^w but the difference between the two growth rates is very small).

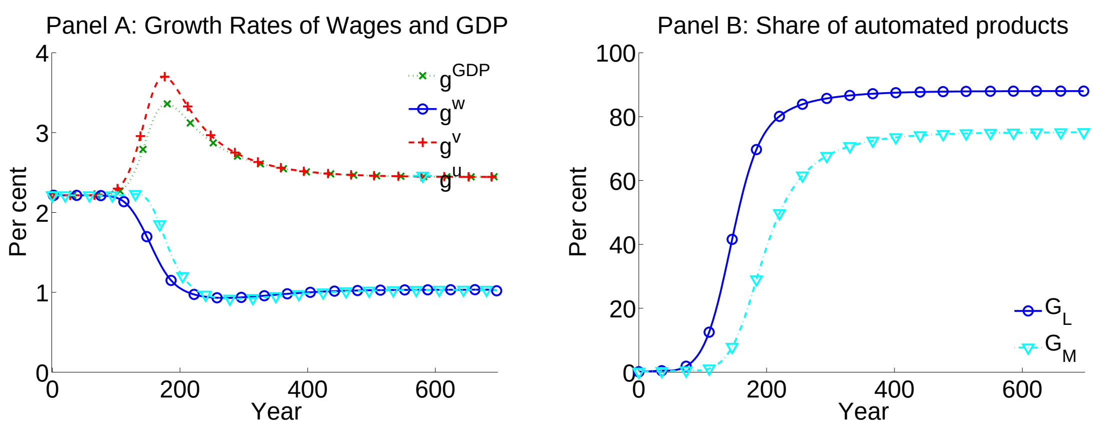

Figure 18: Transitional dynamics with middle-skill workers without an automation externality ($\tilde{\kappa} = 0$).

7.3 Alternative production technology for machines

The assumption of identical production technologies for consumption and machines imposes a constant real price of machines once they are introduced. As shown in Nordhaus (2007) the price of computing power has dropped dramatically over the past 50 years and the declining real price of computers/capital is central to the theories of Autor and Dorn (2013) and Karabarbounis and Neiman (2013). As explained in section 2, it is possible to interpret automation as a decline of the price of a specific equipment from infinity (the machine does not exist) to 1. Yet, our assumption that once a machine is invented, its price is constant, is crucial for deriving the general conditions under which the real wages of low-skill workers must increase asymptotically in Proposition 2. We generalize this in what follows.

Let there be two final good sectors, both perfectly competitive employing CES production technology with identical elasticity of substitution, σ. The output of sector 1, Y, is used for consumption. The output of sector 2, X, is used for machines. The two final good sectors use distinct versions of the same set of intermediate inputs, where we denote the use of intermediate inputs as $y_1(i)$ and $y_2(i)$, respectively, with $i \in [0, N]$. The two versions of intermediate input i are produced by the same intermediate input supplier using production technologies that differ only in the weight on high-skill labor:

$$y_k(i) = \left[l_k(i)^{\frac{\epsilon-1}{\epsilon}} + \alpha(i)(\tilde{\varphi}x_k(i))^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon\beta_k}{\epsilon-1}} h_k(i)^{1-\beta_k},$$

where a subscript, $k = 1, 2$, refers to the sector where the input is used. Importantly, we assume $\beta_2 \geq \beta_1$, such that the production of machines relies more heavily on machines as inputs than the production of the consumption good. Continuing to normalize the price of final good Y to 1, such that the real price of machines is p_t^x, and allowing for the natural extensions of market clearing conditions, we can derive the following generalization of Proposition 2 (where $\psi_k = (\sigma-1)^{-1}(1-\beta_k)^{-1}$).

Proposition 5. *Consider three processes* $[N_t]_{t=0}^{\infty}$, $[G_t]_{t=0}^{\infty}$ *and* $[H_t^P]_{t=0}^{\infty}$ *where* $(N_t, G_t, H_t^P) \in (0,\infty) \times [0,1] \times (0, H]$ *for all* t. *Assume that* G_t, g_t^N *and* H_t^P *all admit strictly positive limits, then:*

$$g_{\infty}^{p^x} = -\psi_2\left(\beta_2 - \beta_1\right) g_{\infty}^N$$

$$g_{\infty}^{GDP} = \left[\psi_1 + \psi_1 \frac{\beta_1\left(\beta_2 - \beta_1\right)}{1 - \beta_2}\right] g_{\infty}^N, \tag{40}$$

and if $G_\infty < 1$ *then the asymptotic growth rate of* w_t *is*[35]

$$g_\infty^w = \frac{1}{1+\beta_1(\sigma-1)} \frac{1-\beta_2+\beta_1\left(\beta_2-\beta_1\right)\left(1-\psi_1^{-1}\right)}{1-\beta_2+\beta_1\left(\beta_2-\beta_1\right)} g_\infty^{GDP}. \tag{41}$$

Proof. See Appendix 8.6. □

Proposition 5 naturally reduces to Proposition 2 for the special case of $\beta_2 = \beta_1$. When $\beta_2 > \beta_1$, the productivity of machine production increases faster than that of the production of Y, implying a gradual decline in the real price of machines. For given g_∞^N, a faster growth in the supply of machines increases the (positive) growth in the relative price of low-skill workers compared with machines, w/p^x, but simultaneously, it reduces the real price of machines, p^x. The combination of these two effects always implies that low-skill workers capture a lower fraction of the growth in Y. Low-skill wages are more likely to fall asymptotically for higher values of the elasticity of substitution between products, σ, as this implies a more rapid substitution away from non-automated products.

7.4 Machines as a capital stock

We assumed so far that machines were an intermediate input that depreciates immediately. In practice, "machines" often take the form of equipment capital, software, etc. which are durable (although their depreciation rate is typically higher than that of structures and housing). We now assume that intermediate inputs producers rent machines from a capital stock. Capital increases with investment and depreciates at a fixed rate, and the investment good is produced with the same technology as the consumption good. Appendix 8.9 derives the equilibrium, here we simply report the results.

Propositions 3 and 2 still holds—with the same sufficient condition (23)—but the system of differential equations must now involve three control variables and three state variables. Moreover, the asymptotic steady state values for the growth rate of the number of products $\left(g^N\right)^*$—and therefore the growth rate of the economy, and the growth rate of low-skill wages—the share of automated products G^*, and the normalized mass of high-skill workers in automation $\widehat{h}^{A*}$ are the same as in the baseline case.

[35]If G_t tends towards 1 sufficiently fast such that $lim_{t\to\infty}(1-G_t)N_t^{\psi_2(1-\mu_1)\frac{\epsilon-1}{\epsilon}}$ is finite, then $g_\infty^w = \frac{1}{\varepsilon}\left(1-\frac{(\beta_2-\beta_1)(\varepsilon-1)}{(1-\beta_2+\beta_1)}\right) g_\infty^{GDP} \geq g_\infty^{p^x}$ whether ϵ is finite or not. It is clear that there always exists an ϵ sufficiently high for the real wage of low-skill workers to decline asymptotically.

Figure 19 shows the transitional dynamics when machines are a capital stock. We choose $G_0 = 0.02$, a depreciation rate $\Delta = 0.1$, an initial capital stock $K_0 = 5.25 \times 10^{-4}$ (chosen such that the initial interest rate is the same as in a world with no automation), all the other parameters are identical to the baseline case. The transitional dynamics look similar to the baseline case, but the central variable which determines whether automation is intensive or not, is now the ratio of low-skill wages to the gross rental rate of capital (therefore Phase 2 occurs sooner). In addition, as shown in Panel D, since the expenditures on machines now correspond to capital income, the decline in the labor share tends to be more pronounced in this case, and high-skill workers' income need not become a larger share of GDP.

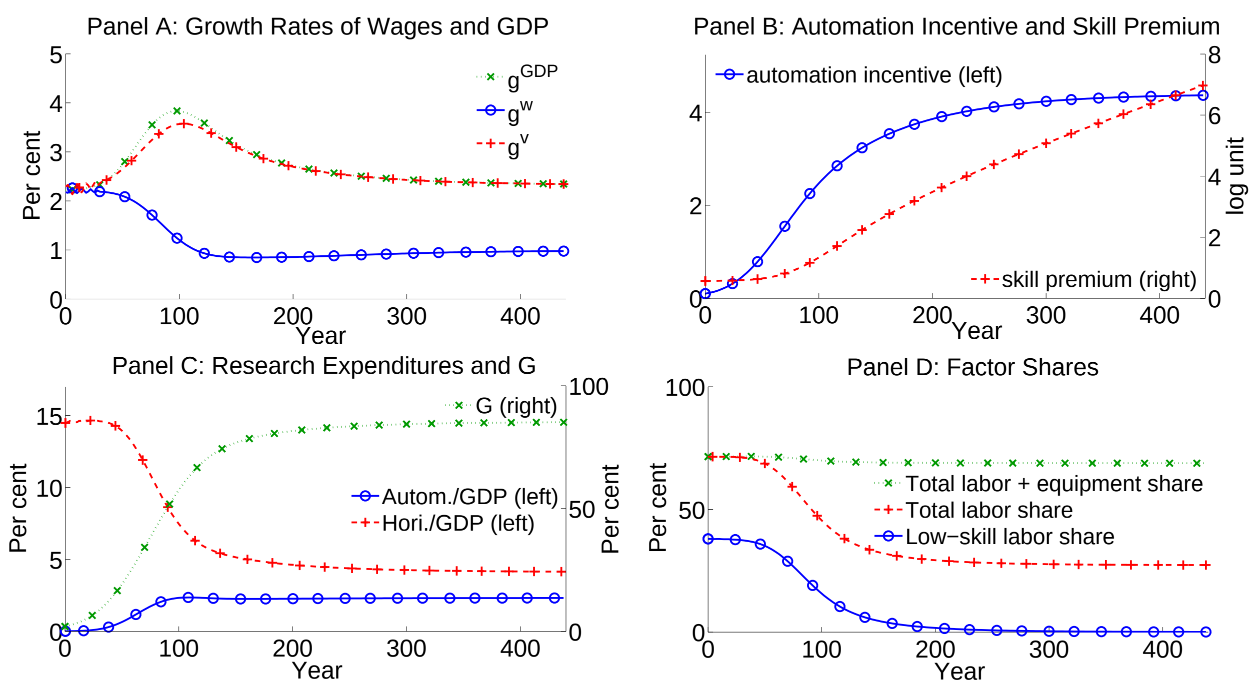

Figure 19: Transitional dynamics when machines are a capital stock.

8 Secondary Appendix (For Online Publication)

8.1 Relationship between wages and N and G

8.1.1 Imperfect substitute case: $\epsilon < \infty$.

We first focus on the imperfect substitute case. Rewrite (10) as

$$\frac{v}{w} = \frac{1-\beta}{\beta} \frac{L}{H^P} \frac{G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu} + (1-G)}{G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1} + (1-G)}. \tag{42}$$

Since $0 < \mu < 1$, (10) establishes v as a function of G, H^P and w (but not N) such that v is increasing in w, $v/w > (1-\beta)/\beta \times L/H^P$ for $G > 0$, and v is increasing in G and decreasing in H^P. The productivity condition similarly establishes v as a function of N, G and w (but not H^P), v is decreasing in w and increasing in N and G. It is then immediate that v, w are jointly uniquely determined by (10) and (11) for given N, G and H^P, both increase in N, and v increases in G (in addition, since (11) traces an iso-cost curve in the input prices plan, the productivity condition curve is convex). We now analyze how w changes with G (for given N and H^P).

To do so, we combine both equations to get:

$$w = \begin{array}{c} \frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)} N^{\frac{1}{\sigma-1}} \left(G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1} + (1-G)\right)^{1-\beta} \\ \times \left(G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu} + (1-G)\right)^{\frac{1}{\sigma-1}-(1-\beta)} \end{array}, \tag{43}$$

(which is the same as equation (39) with different notations). For given N and H^P, (43) defines w as an implicit function of G, $W(G)$. Further:

$$W(0) = \frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)} N^{\frac{1}{\sigma-1}} \text{ so that } W(1) = W(0)\left(1+\varphi w^{\varepsilon-1}\right)^{\frac{\mu}{\sigma-1}-(1-\beta)},$$

therefore $W(1) > W(0)$ if and only if $\frac{\mu}{\sigma-1} - (1-\beta)$ (that is $\frac{\beta}{1-\beta} > \varepsilon - 1$) as claimed in the text.

Define $\psi \equiv 1/\left((1-\beta)[\sigma-1]\right)$ and differentiate (43) to obtain

$$W'(G) = \frac{(1-\beta)\, w W^{num}}{W^{den}},$$

with

$$W^{den}(w,G) = \begin{array}{c} 1-\beta\frac{G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}}{G(1+\varphi w^{\varepsilon-1})^{\mu}+(1-G)}\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}}+(1-\beta)(\varepsilon-1) \\ \times\left(\frac{(1-\mu)G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1}}{G(1+\varphi w^{\varepsilon-1})^{\mu-1}+(1-G)}+\frac{\mu G\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}}{G(1+\varphi w^{\varepsilon-1})^{\mu}+(1-G)}\right)\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}} \end{array},$$

and

$$W^{num}(w,G) = \frac{(1+\varphi w^{\varepsilon-1})^{\mu-1}-1}{G(1+\varphi w^{\varepsilon-1})^{\mu-1}+(1-G)} + (\psi-1)\frac{\left((1+\varphi w^{\varepsilon-1})^{\mu}-1\right)}{G(1+\varphi w^{\varepsilon-1})^{\mu}+(1-G)}.$$

W^{den} is always strictly positive, therefore W' has the sign of W^{num}. $W^{num}<0$ if $\psi\leq 1$ since $(1+\varphi w^{\varepsilon-1})^{\mu-1}<1$ and $(1+\varphi w^{\varepsilon-1})^{\mu}>1$, therefore for $(1-\beta)(\sigma-1)\geq 1$, w is decreasing in G.

Case where $1<\psi\leq\mu^{-1}$. Assume that this is the case. Note that

$$W^{num}(w,1) = \frac{(1+\varphi w^{\varepsilon-1})^{\mu-1}-1}{(1+\varphi w^{\varepsilon-1})^{\mu-1}} + (\psi-1)\frac{\left((1+\varphi w^{\varepsilon-1})^{\mu}-1\right)}{(1+\varphi w^{\varepsilon-1})^{\mu}},$$

so that

$$\begin{aligned} W^{num}(w,1) \underset{w\to 0}{\sim} (\psi\mu-1)\varphi w^{\varepsilon-1} \text{ if } \psi &\neq \mu^{-1}, \\ W^{num}(w,1) \underset{w\to 0}{\sim} -\frac{(1-\mu)}{2}\left(\varphi w^{\varepsilon-1}\right)^{2} \text{ if } \psi &= \mu^{-1}, \end{aligned} \tag{44}$$

and we have $W^{num}(w,1)<0$ for w low enough. In addition

$$\frac{dW^{num}(w,1)}{dw} = -\left(1-\mu\psi+(1-\mu)\varphi w^{\varepsilon-1}\right)\left(1+\varphi w^{\varepsilon-1}\right)^{-\mu-1}\varphi(\varepsilon-1)w^{\varepsilon-2}<0, \tag{45}$$

since $1-\mu>0$ and $1-\mu\psi\geq 0$. Therefore we always have $W^{num}(w,1)<0$, in particular this means that $W^{num}(W(1),1)<0$, so that W is a decreasing function of G for G close to 1.

Suppose that W is not everywhere decreasing in G, then there must exist G^{peak} such that $W^{num}\left(W\left(G^{peak}\right),G^{peak}\right)=0$. This implies that

$$G^{peak} = \frac{-\psi^{-1}\varphi w^{\varepsilon-1}(1+\varphi w^{\varepsilon-1})^{\mu-1}+\left((1+\varphi w^{\varepsilon-1})^{\mu}-1\right)}{\left(1-(1+\varphi w^{\varepsilon-1})^{\mu-1}\right)\left((1+\varphi w^{\varepsilon-1})^{\mu}-1\right)},$$

Defining

$$T\left(w\right)\equiv w^{1-\frac{\varepsilon-1}{\sigma-1}}\left(\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}-1\right)^{1-\beta}\left(1-\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1}\right)^{\frac{1-\psi^{-1}}{\sigma-1}}\left(1+\varphi w^{\varepsilon-1}\right)^{\frac{1-\mu}{\sigma-1}},$$

and G^{peak} into (43), we obtain that $W^{num}\left(W\left(G\right),G\right)=0$ requires:

$$T\left(w\right)=\frac{\sigma-1}{\sigma}\beta\left(\frac{H^{P}}{L}\right)^{(1-\beta)}\left(\psi^{-1}\right)^{1-\beta}\left(1-\psi^{-1}\right)^{\frac{1}{\sigma-1}-(1-\beta)}\left(\varphi N\right)^{\frac{1}{\sigma-1}}.$$

We are now going to show that $T\left(w\right)$ is a monotone function of w. Since $\psi>1$, and $\mu<1$, then T is increasing in w whenever $\frac{\varepsilon-1}{\sigma-1}<1$ (that is $\varepsilon\leq\sigma$). Assume now that $\varepsilon>\sigma$, we can derive:

$$\frac{d\ln T\left(w\right)}{d\ln w}=\begin{array}{c}1-\frac{\varepsilon-1}{\sigma-1}+\left(1-\beta\right)\mu\left(\varepsilon-1\right)\frac{\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}}{\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}-1}\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}}+\frac{1-\psi^{-1}}{\sigma-1}\left(1-\mu\right)\\ \times\left(\varepsilon-1\right)\frac{\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1}}{1-\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1}}\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}}+\frac{1-\mu}{\sigma-1}\left(\varepsilon-1\right)\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}}\end{array}.$$

Rearranging terms we can derive:

$$\frac{d\ln T\left(w\right)}{d\ln w}=\frac{\left\{\begin{array}{c}\left(1-\beta\right)\left(\varepsilon-1\right)\left(\left(1+\mu\varphi w^{\varepsilon-1}\right)\left(1+\varphi w^{\varepsilon-1}\right)^{-\mu}-1\right)\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}}\\ +\left(\frac{\varepsilon-1}{\sigma-1}-1\right)\left(1-\left(1+\varphi w^{\varepsilon-1}\right)^{-\mu}\right)^{2}\\ +\left(1-\beta\right)\left(1-\left(1+\varphi w^{\varepsilon-1}\right)^{-\mu}\right)\left(1+\varphi w^{\varepsilon-1}\right)^{1-\mu}\frac{\varphi w^{\varepsilon-1}}{1+\varphi w^{\varepsilon-1}}\end{array}\right\}}{\left(1-\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1}\right)\left(\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}-1\right)}.$$

In this expression, the denominator is positive, the second term and the third term in the numerator are also positive. To determine the sign of the first term note that the function $q\left(x\right)\equiv\left(1+x\right)^{-\mu}\left(1+\mu x\right)$ is increasing for $x\geq0$ whenever $\mu\in\left(0,1\right)$ as $q'\left(x\right)=\left(1-\mu\right)\mu x\left(1+x\right)^{-\mu-1}$, therefore since $q\left(0\right)=1$, the first term in the numerator is also positive. As a result $T\left(w\right)$ is a monotone function, which means that there is only one value of w consistent with $W'\left(G^{peak}\right)=0$. As a result, W' changes sign at most once.

Therefore if $W\left(G\right)$ is not decreasing everywhere, it must be increasing for $G=0$. We can derive

$$W^{num}\left(w,0\right)=\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-1}+\left(\psi-1\right)\left(1+\varphi w^{\varepsilon-1}\right)^{\mu}-\psi.$$

Note that

$$W^{num}(w,0) \underset{w\to 0}{\sim} (\mu\psi - 1)\,\varphi w^{\varepsilon-1} \text{ if } \psi \neq \mu^{-1} \tag{46}$$
$$W^{num}(w,0) \underset{w\to 0}{\sim} (1-\mu)\,\frac{\left(\varphi w^{\varepsilon-1}\right)^2}{2} \text{ if } \psi = \mu^{-1}.$$

Therefore, if w is low enough, $W^{num}(w,0) < 0$ if $\psi < \mu^{-1}$ (but it is positive if $\psi = \mu^{-1}$). In addition, we have

$$\frac{dW^{num}(w,0)}{dw} = \left(\psi\mu - 1 + (\psi - 1)\,\mu\varphi w^{\varepsilon-1}\right)\varphi w^{\varepsilon-2}\left(1+\varphi w^{\varepsilon-1}\right)^{\mu-2}, \tag{47}$$

When $\psi\mu = 1$, $\frac{dW^{num}(w,0)}{dw} > 0$, so that $W'(0) > 0$, which implies that W is an inversely u-shaped function of G (initially increasing and then decreasing).

When $\psi < \mu^{-1}$, $\frac{dW^{num}(w,0)}{dw} < 0$ for $w < \left(\frac{1-\psi\mu}{\mu\varphi(\psi-1)}\right)^{\frac{1}{\varepsilon-1}}$ and positive otherwise, in addition $W^{num}(w,0) < 0$ for w close to 0 and $W^{num}(w,0)$ is positive for w large enough (as $\psi > 1$). This implies that $W^{num}(w,0)$ is negative for w below a threshold value and positive if w is greater than that threshold. Therefore $W'(0)$ is negative if $W(0)$ is below a threshold value and positive otherwise, since $W(0)$ monotonically increases with N, the same statement holds replacing $W(0)$ by N. Therefore we get that when N is low enough W is everywhere decreasing in G, while for N high enough W is inversely u-shaped.

Case where $\psi > \mu^{-1}$. The reasoning on the possibility for a solution to $W^{num}(W(G), G) = 0$ still applies, therefore $W'(G)$ can change sign at most once, since $W(1) > W(0)$ in this case, we know that W must be increasing on some interval. (46) and (47) imply that when $\psi > \mu^{-1}$, $W^{num}(w,0)$ is always positive. (44) and (45) imply that $W'(1)$ is negative if and only if $W(1)$ is large enough (which is equivalent to N large enough).

8.1.2 Perfect substitute case:$\epsilon = \infty$

In the perfect substitute case, there are three possibilities: i) either $w < \widetilde{\varphi}^{-1}$ in which case automated firms only use low-skill workers and low-skill wages are given by

$$w = \frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)} N^{\frac{1}{\sigma-1}}, \tag{48}$$

with a skill premium obeying $\frac{v}{w}=\frac{1-\beta}{\beta}\frac{L}{H^P}$; ii) or $w=\widetilde{\varphi}^{-1}$ and automated firms use machines but also possibly workers, in which case high-skill wages can be obtained from (11) which is now written as:

$$\frac{\sigma}{\sigma-1}\frac{N^{\frac{1}{1-\sigma}}}{\beta^{\beta}\left(1-\beta\right)^{1-\beta}}\widetilde{\varphi}^{-\beta}v^{1-\beta}=1;$$

iii) or finally, $w>\widetilde{\varphi}^{-1}$ and all automated firms use machines only, in that case, we get that (43) is replaced by

$$w=\frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)}N^{\frac{1}{\sigma-1}}\left(1-G\right)^{1-\beta}\left(G\left(\widetilde{\varphi}w\right)^{\beta(\sigma-1)}+1-G\right)^{\frac{1}{\sigma-1}-(1-\beta)}, \qquad (49)$$

and the skill premium obeys:

$$\frac{v}{w}=\frac{1-\beta}{\beta}\frac{L}{H^P}\frac{G+\left(1-G\right)\left(w\widetilde{\varphi}\right)^{-\beta(\sigma-1)}}{\left(1-G\right)\left(w\widetilde{\varphi}\right)^{-\beta(\sigma-1)}}.$$

It is direct to show that (49) defines w uniquely as a function of N,G and H^P, with w increasing in N and in H^P. In addition, w as defined by (49) follows the pattern of Proposition 1 ii) in terms of comparative statics with respect to G, with $w=\frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)}N^{\frac{1}{\sigma-1}}$ for $G=0$ and $w=0$ for $G=1$ (this is for w as defined by (49) and not for the equilibrium w).

Considering that (49) only applies when $w>\widetilde{\varphi}^{-1}$ and looking at the three different cases together; we obtain that w is initially increasing in N (up until $\frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)}N^{\frac{1}{\sigma-1}}=\widetilde{\varphi}^{-1}$), it is then constant equal to $\widetilde{\varphi}^{-1}$ (up until $\frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)}N^{\frac{1}{\sigma-1}}=\widetilde{\varphi}^{-1}/\left(1-G\right)^{1-\beta}$) and increasing thereafter (if $G=1$ then this last part does not exist and w stays at $\widetilde{\varphi}^{-1}$).

Similarly, for low N (that is $\frac{\sigma-1}{\sigma}\beta\left(\frac{H^P}{L}\right)^{(1-\beta)}N^{\frac{1}{\sigma-1}}\leq\widetilde{\varphi}^{-1}$), w is independent of G. Then the pattern of ii) follows with w either decreasing or inversely u-shaped in G except that w will be constant over an interval of the type $[G^{cons},1]$ for some $G^{cons}\in[0,1)$.

High-skill wages are always increasing in N and G.

8.2 Proofs of the asymptotic results

8.2.1 Proof of Proposition 2

To see that w_t is bounded from below, assume that liminfw_t=0. Then using that $H_t^P$and G_t admit positive limits, (10) implies that liminf$v_t = 0$. Plugging this further in (11) gives liminf$N_t = 0$, which is impossible since g_t^N admits a positive limit. Therefore, w_t must be bounded below, so that (11) gives $g_\infty^v = \psi g_\infty^N$. Further, using that H_t^P admits a limit and (8) gives the growth rate of Y_t. We now derive the asymptotic growth rate of w_t. To do so we consider in turn the case where $\epsilon < \infty$, and the case where $\epsilon = \infty$.

Case with $\epsilon < \infty$. We use equation (43) which gives w_t as a function of N_t, G_t and H_t^P. Note that assuming that $w_t$is bounded above leads to a contradiction, therefore lim$w_t = \infty$.

Assume first that $G_\infty < 1$, then, since $\lim w_t = \infty$, (43) implies

$$w_t \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}} (1-G_\infty)\frac{H_\infty^P}{L}(G_\infty\varphi^\mu)^{\psi-1}\right)^{\frac{1}{1+\beta(\sigma-1)}} N_t^{\frac{\psi}{1+\beta(\sigma-1)}},$$

where for x_t and y_t (possibly with no limits), $x_t \sim y_t$ signifies $x_t/y_t \to 1$. This delivers Part A).

Consider now the case where $G_\infty = 1$. Note that (43) gives:

$$w_t \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}} \frac{H_\infty^P}{L}\varphi^{\mu(\psi-1)}\right)^{\frac{1}{\varepsilon}} N_t^{\frac{\psi}{\varepsilon}} \left(\varphi^{\mu-1} + (1-G_t)\, w_t^{(\varepsilon-1)(1-\mu)}\right)^{\frac{1}{\varepsilon}}. \tag{50}$$

Following the assumption of Part B in Proposition 2, we assume that $\lim\,(1-G_t)\, N_t^{\frac{\psi}{\varepsilon}(\varepsilon-1)(1-\mu)}$ exists and is finite. Suppose first that $\limsup\,(1-G_t)\, w_t^{(\varepsilon-1)(1-\mu)} = \infty$, then there must exist a sequence of t's, denoted t_n for which:

$$w_{t_n} \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}} \frac{H_\infty^P}{L}(\varphi^\mu)^{\psi-1}\right)^{\frac{1}{1+\beta(\sigma-1)}} \left((1-G_{t_n})\, N_{t_n}^{\psi}\right)^{\frac{1}{1+\beta(\sigma-1)}}.$$

Yet, this implies

$$\left(1-G_{t_n}\right)w_{t_n}^{(\varepsilon-1)(1-\mu)} \sim \begin{array}{l}\left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}}\frac{H_\infty^P}{L}\left(\varphi^\mu\right)^{\psi-1}\right)^{\frac{(\varepsilon-1)(1-\mu)}{1+\beta(\sigma-1)}} \\ \times\left(\left(1-G_{t_n}\right)N_{t_n}^{\frac{\psi(\varepsilon-1)(1-\mu)}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\beta(\sigma-1)}}\end{array},$$

the left-hand side is assumed to be unbounded, while the right-hand side is bounded: there is a contradiction. Therefore, $\limsup\left(1-G_t\right)w_t^{(\varepsilon-1)(1-\mu)}<\infty$.

Consider now the possibility that $\lim\left(1-G_t\right)w_t^{(\varepsilon-1)(1-\mu)}=0$, then (50) implies

$$w_t \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}}\frac{H_\infty^P}{L}\varphi^{\mu\psi-1}\right)^{\frac{1}{\varepsilon}}N_t^{\frac{\psi}{\varepsilon}},$$

which delivers that g_t^w exists and is given by $\frac{\psi}{\varepsilon}g_t^N=\frac{1}{\varepsilon}g_t^Y$.

Alternatively, $\limsup\left(1-G_t\right)w_t^{(\varepsilon-1)(1-\mu)}$ is finite but strictly positive (given by λ_1). In this case, there exists a sequence of $t's$, denoted t_m such that

$$w_{t_m} \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}}\frac{H_\infty^P}{L}\varphi^{\mu(\psi-1)}\left(\varphi^{\mu-1}+\lambda_1\right)\right)^{\frac{1}{\varepsilon}}N_{t_m}^{\frac{\psi}{\varepsilon}}. \tag{51}$$

This leads to

$$\lambda_1 \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}}\frac{H_\infty^P}{L}\varphi^{\mu(\psi-1)}\left(\varphi^{\mu-1}+\lambda_1\right)\right)^{\frac{(\varepsilon-1)(1-\mu)}{\varepsilon}}\left(1-G_{t_m}\right)N_{t_m}^{\frac{\psi}{\varepsilon}(\varepsilon-1)(1-\mu)},$$

which is only possible if $\lim\left(1-G_t\right)N_t^{\frac{\psi}{\varepsilon}(\varepsilon-1)(1-\mu)}>0$. We denote such a limit by λ. Then (50) leads to

$$\left(w_t^\varepsilon N_t^{-\psi}\right) \sim \left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}}\frac{H_\infty^P}{L}\varphi^{\mu(\psi-1)}\left(\varphi^{\mu-1}+\lambda\left(N_t^{-\psi}w_t^\varepsilon\right)^{\frac{(\varepsilon-1)(1-\mu)}{\varepsilon}}\right),$$

which defines uniquely the limit of $w_t^\varepsilon N_t^{-\psi}$. Therefore g_t^w admits a limit, which here too is given by $g_t^w=\frac{\psi}{\epsilon}g_t^N$. This completes the poof of part B).

To prove footnote 7, assume now that, $\lim\left(1-G_t\right)N_t^{\frac{\psi}{\varepsilon}(\varepsilon-1)(1-\mu)}=\infty$. Consider further the case where $\liminf\left(1-G_t\right)w_t^{(\varepsilon-1)(1-\mu)}=\lambda_2<\infty$, then there is a sequence of t's denoted t_μ for which (51) applies with t_μ replacing t_m and λ_2 replacing λ_1. Then following the same steps as above, we get that $\liminf \left(1 - G_t\right) N_t^{\frac{\psi}{\varepsilon}(\varepsilon-1)(1-\mu)}$ is finite, which is a direct contradiction. Therefore, we must have $\lim \left(1 - G_t\right) w_t^{(\varepsilon-1)(1-\mu)} = \infty$. In which case, (50) directly implies

$$w_t \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\right)^{\frac{1}{1-\beta}} \frac{H_\infty^P}{L} \varphi^{\mu(\psi-1)}\right)^{\frac{1}{1+\beta(\sigma-1)}} \left(\left(1 - G_t\right) N_t^{\psi}\right)^{\frac{1}{1+\beta(\sigma-1)}}.$$

Assume that the growth rate of $\left(1 - G_t\right) N_t^{\frac{\psi}{\varepsilon}(\varepsilon-1)(1-\mu)}$ admits a limit g_λ (so that $0 \leq g_\lambda \leq \frac{\psi}{\varepsilon}\left(\varepsilon - 1\right)\left(1 - \mu\right) g_\infty^N$), then g_t^w must also admit a limit, and we have

$$g_t^w = \frac{1}{1 + \beta\left(\sigma - 1\right)} \left(g_\lambda + \left(1 - \frac{\left(\varepsilon - 1\right)}{\varepsilon}\left(1 - \mu\right)\right)\psi g_t^N\right),$$

which directly leads to $\frac{1}{\varepsilon} g_\infty^{GDP} \leq g_\infty^w \leq \frac{1}{1+\beta(\sigma-1)} g_\infty^{GDP}$.

Case with $\epsilon = \infty$. Low skill wages are now defined as described in Appendix 8.1.2. First consider the case where $G_\infty < 1$, then Part A) immediately follows. Assume now that $G_\infty = 1$ and that $\lim \left(1 - G_t\right) N_t^{\psi}$ exists and is finite. Note first that (48) implies that w_t must be bounded weakly above $\widetilde{\varphi}$ in the long-run. As a result, (49) leads to

$$w_t \sim \left(\left(\frac{\sigma-1}{\sigma}\beta\widetilde{\varphi}^{\beta\left(1-\psi^{-1}\right)}\right)^{\frac{1}{1-\beta}} \frac{H_\infty^P}{L}\right)^{\frac{1}{1+\beta(\sigma-1)}} \left(\left(1 - G_t\right) N_t^{\psi}\right)^{\frac{1}{1+\beta(\sigma-1)}} \text{ if } w_t > \widetilde{\varphi}.$$

Since $\lim \left(1 - G_t\right) N_t^{\psi}$ exists and is finite, w_t also admits a finite limit. In particular, if $\lim \left(1 - G_t\right) N_t^{\psi} = 0$, then $w_\infty = \widetilde{\varphi}$.

Skill premium. An increase in N increases w, which from lemma 1 implies that the skill premium v/w increases as long as $G > 0$. Consider the case where w decreases with G, then since v increases with G, the skill premium must increase with G. Consider now the case where w increases with G, then from equation (10), v/w must increase both through the direct impact of G and through the indirect impact coming from the increase in w.

8.2.2 Proof of Lemma 2

Note that $G_t N_t$ is the mass of automated firms and let $\nu_{1,t} > 0$ be the intensity at which non-automated firms are automated at time t and $0 \leq \nu_{2,t} < 1$ be the fraction of new products introduced at time t that are initially automated. Then $\dot{\left(G_t N_t\right)} =$

$\nu_{1,t}(1-G_t)N_t + \nu_{2,t}\dot{N}_t$ such that $\dot{G}_t = \nu_{1,t}(1-G_t) - (G_t - \nu_{2,t})g_t^N$. First assume that $G_\infty = 1$, then if $\nu_{1,t} < \bar{\nu}_1 < \infty$ and $\nu_{2,t} < \bar{\nu}_2 < 1$, we get that $\dot{G}_t$ must be negative for sufficiently large t, which contradicts the assumption that $G_\infty = 1$. Similarly if $G_\infty = 0$, then having $\nu_{1,t} > \underline{\nu}$ for all t, gives that $\dot{G}_t$ must be positive for sufficiently large t, which also implies a contradiction. Hence a limit must have $0 < G_\infty < 1$.

8.3 Proofs and analytical results for the baseline dynamic model

8.3.1 Proof of Proposition 3

We look for a steady state with positive long-run growth for the system defined by (27), (28), (30) and (31), and we denote such a (potential) steady state $n^*, G^*, \widehat{h}^{A*}, \chi^*$ (more generally we denote all variables at steady state with a *).[36] Following (27), we immediately get that $n^* = 0$. Equation (39), implies that $\omega^* = 0$ (recall that $\mu \in (0,1)$). As a result, using (35), (31) implies that in steady state,

$$\widehat{h}^{A*} = \frac{\kappa}{\gamma(1-\kappa)}\left(\rho + ((\theta-1)\psi + 1)\, g^{N*}\right) \tag{52}$$

which uniquely defines defines $\widehat{h}^{A*}$ as a linear and increasing function of g^{N*} (recall that $\theta \geq 1$). Note that if $g^{N*} > 0$, then $\widehat{h}^{A*} > 0$. Using (28), we get that G^* obeys:

$$G^* = \frac{\eta\,(G^*)^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}}{\eta\,(G^*)^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa} + g^{N*}}. \tag{53}$$

For $G^* > 0$, this equation, combined with (52), defines G^* uniquely as an increasing function of g^{N*}, and, with $\left(g^N\right)^* > 0$, $G^* < 1$ (when $\widetilde{\kappa} > 0$, $G^* = 0$ is also a solution, but, here, we are focusing on a steady state with a strictly positive G^*). Note that (38) also uniquely defines H^{P*} as a function of g^{N*}:

$$H^{P*} = H - \frac{g^{N*}}{\gamma} - (1-G^*)\,\widehat{h}^{A*}. \tag{54}$$

[36] Note that if $g_t^N = 0$, the economy does not obey this system of equations but that it is also impossible to achieve positive long-run growth, as production is bounded by the production of an economy which has $G_t = 1$.

Using $\omega^* = 0$, (35) and (53) allows to rewrite (30) in steady state as:

$$\frac{\eta\kappa \left(G^*\right)^{\widetilde{\kappa}-1} \left(\widehat{h}^{A*}\right)^{\kappa}}{1-\kappa} \psi H^{P*} = \frac{\gamma}{\kappa} \left(\widehat{h}^{A*}\right)^{2} + \eta G_t^{\widetilde{\kappa}} \left(\widehat{h}^{A*}\right)^{\kappa+1}. \tag{55}$$

Since G^*, $\widehat{h}^{A*}$ and H^{P*} are functions of g^{N*}, one can rewrite (55) as an equation determining g^{N*}. A steady state with positive growth-rate is a solution to

$$f\left(g^{N*}\right) \equiv \frac{1-\kappa}{\kappa} \frac{\gamma G^* \widehat{h}^{A*}}{\psi H^{P*}} \left(\frac{1}{\kappa\eta \left(G^*\right)^{\widetilde{\kappa}}} \left(\widehat{h}^{A*}\right)^{1-\kappa} + \frac{1}{\gamma} \right) = 1, \tag{56}$$

with $g^{N*} > 0$. Indeed, (37) simply determines χ^* as:

$$\chi^* = \left(\frac{\sigma}{\sigma-1}\right)^{\frac{1}{1-\beta}(1-\theta\beta)} \frac{\left(1-\beta\frac{\sigma-1}{\sigma}\right)^{\theta} \left(H^{P*}\right)^{\theta}}{(1-\beta)\, \beta^{\frac{\beta}{1-\beta}(1-\theta)} \left(G^* \varphi^{\mu}\right)^{\psi(1-\theta)}}, \tag{57}$$

which achieves the characterization of a steady state for the system of differential equations defined by (27), (28), (30) and (31).

In order to establish the sufficiency of equation (23) for positive growth. Note that as $g^{N*} \to 0$, then equations (52), (53) and (54) imply that

$$f(0) = \frac{\rho}{\psi H} \left(\frac{1}{\eta\kappa^{\kappa} (1-\kappa)^{1-\kappa}} \left(\frac{\rho}{\gamma}\right)^{1-\kappa} + \frac{1}{\gamma} \right).$$

In addition, $\frac{g^{N*}}{\gamma} + (1-G^*)\, \widehat{h}^{A*}$ is always greater than $\frac{g^{N*}}{\gamma}$, therefore for a sufficiently large g^{N*} (smaller than γH), H^{P*} is arbitrarily small, while for the same value G^* and $\widehat{h}^{A*}$ are bounded below and above. This establishes that for g^{N*} large enough, $f\left(g^{N*}\right) > 1$. Therefore a sufficient condition for the existence of at least one steady state with positive growth and positive G^* is that $f(0) < 1$ (such that $f\left(g^{N*}\right) = 1$ has a solution), which is equivalent to condition (23).

8.3.2 Uniqueness of the steady state

Generally the steady state is not unique. Nonetheless, consider the special case in which $\widetilde{\kappa} = 0$. Then f can be rewritten as

$$f\left(g^{N*}\right) = \frac{1-\kappa}{\kappa} \frac{\gamma G^* \widehat{h}^{A*}}{\psi H^{P*}} \left(\frac{1}{\kappa\eta} \left(\widehat{h}^{A*}\right)^{1-\kappa} + \frac{1}{\gamma} \right), \tag{58}$$

note that H^{P*} is decreasing in g^{N*} and $\widehat{h}^{A*}$ is increasing in g^{N*}, so a sufficient condition for f to be increasing in g^{N*} is that $G^*\widehat{h}^{A*}$ is also increasing in g^{N*}. With $\widetilde{\kappa}=0$, using (52), (53), we get:

$$G^*\widehat{h}^{A*}=\frac{\eta\left(\frac{\kappa}{\gamma(1-\kappa)}\right)^{\kappa+1}\left(\rho+\left((\theta-1)\,\psi+1\right)g^{N*}\right)^{\kappa+1}}{\eta\left(\frac{\kappa}{\gamma(1-\kappa)}\right)^{\kappa}\left(\rho+\left((\theta-1)\,\psi+1\right)g^{N*}\right)^{\kappa}+g^{N*}}.$$

Therefore

$$\frac{d\left(G^*\widehat{h}^{A*}\right)}{dg^{N*}}=\begin{array}{c}\frac{\eta\left(\frac{\kappa}{\gamma(1-\kappa)}\right)^{\kappa+1}\left(\rho+((\theta-1)\psi+1)g^{N*}\right)^{\kappa}}{\left(\eta\left(\frac{\kappa}{\gamma(1-\kappa)}\right)^{\kappa}\left(\rho+((\theta-1)\psi+1)g^{N*}\right)^{\kappa}+g^{N*}\right)^{2}}\\ \times\left(\begin{array}{c}\eta\left(\frac{\kappa}{\gamma(1-\kappa)}\right)^{\kappa}\left((\theta-1)\,\psi+1\right)\left(\rho+\left((\theta-1)\,\psi+1\right)g^{N*}\right)^{\kappa}\\ -\rho+g^{N*}\kappa\left((\theta-1)\,\psi+1\right)\end{array}\right)\end{array}.$$

Since $g^{N*}>0$, we get that $\frac{d\left(G^*\widehat{h}^{A*}\right)}{dg^{N*}}>0$ (so that the steady state is unique) if $\frac{(1-\kappa)^{\kappa}\gamma^{\kappa}}{\eta\kappa^{\kappa}}\rho^{1-\kappa}<(\theta-1)\,\psi+1$. This condition is likely to be met for reasonable parameter values as long as the automation technology is not too concave: ρ is a small number, $\theta\geq 1$ and γ and η being innovation productivity parameters should be of the same order (it is indeed met for our baseline parameters).

8.3.3 Transitional dynamics and the first phase

Combining (24) and (25), we can write:

$$N_t h_t^A=\left(\kappa\eta G_t^{\widetilde{\kappa}}\left(\int_t^{\infty}\exp\left(-\int_t^{\tau}r_u du\right)\left(\frac{N_t}{v_t}\left(\pi_\tau^A-\pi_\tau^N\right)d\tau-\frac{1-\kappa}{\kappa}\frac{N_t}{N_\tau}\frac{v_\tau}{v_t}\left(N_\tau h_\tau^A\right)\right)d\tau\right)\right)^{\frac{1}{1-\kappa}}.$$

Using (8) and that aggregate profits $\Pi_t=N_t\left(G_t\pi_t^A+(1-G_t)\,\pi_t^N\right)$ are a share $1/\sigma$ of output, we can rewrite this equation as:

$$\widehat{h}_t^A=\left(\kappa\eta G_t^{\widetilde{\kappa}}\left(\int_t^{\infty}\exp\left(-\int_t^{\tau}r_u du\right)\left(\psi H_t^P\frac{\pi_\tau^A-\pi_\tau^N}{G_t\pi_t^A+(1-G_t)\,\pi_t^N}d\tau-\frac{1-\kappa}{\kappa}\frac{N_t}{N_\tau}\frac{v_\tau}{v_t}\widehat{h}_\tau^A\right)d\tau\right)\right)^{\frac{1}{1-\kappa}}. \tag{59}$$

Recalling (7), we can write:

$$\widehat{h}_t^A = \left(\kappa\eta G_t^{\widetilde{\kappa}}\left(\int_t^{\infty}\left(\begin{array}{c}\psi H_t^P \frac{\left(1+\varphi w_\tau^{\epsilon-1}\right)^\mu-1}{G_t\left(1+\varphi w_t^{\epsilon-1}\right)^\mu+1}\exp\left(\int_t^\tau\left(g_u^{\pi^N}-r_u\right)du\right)\\ -\frac{1-\kappa}{\kappa}\exp\left(\int_t^\tau\left(g_u^v-g_u^N-r_u\right)du\right)\widehat{h}_\tau^A\end{array}\right)d\tau\right)\right)^{\frac{1}{1-\kappa}}.$$

Consider a fixed $\widehat{t}>0$. Then for an arbitrarily large T, if w_0 is sufficiently small relative to $\widetilde{\varphi}^{-1}$, we will have that w_t is small relative to $\widetilde{\varphi}^{-1}$ over $\left(0,\widehat{t}+T\right)$. For any $\tau\in\left(0,\widehat{t}+T\right)$, we have that $\frac{\left(1+\varphi w_\tau^{\epsilon-1}\right)^\mu-1}{G_t\left(1+\varphi w_t^{\epsilon-1}\right)^\mu+1}=\mu\varphi w_\tau^{\epsilon-1}+o\left(\varphi w_\tau^{\epsilon-1}\right)$. The notation $o(z)$ denotes negligible relative to z (that is $f(z)=o(z)$, if $\lim\limits_{z\to 0}f(z)/z=0$) and $O(z)$ will denote of the same order or negligible in front of z ($f(z)=O(z)$ if $\limsup\limits_{z\to 0}|f(z)/z|<\infty$). Then for any $t\in\left(0,\widehat{t}\right)$

$$\left(\widehat{h}_t^A\right)^{1-\kappa}\leq\kappa\eta G_t^{\widetilde{\kappa}}\left(\begin{array}{c}\int_t^{\widehat{t}+T}\psi H_t^P\left(\mu\varphi w_\tau^{\epsilon-1}+o\left(\varphi w_\tau^{\epsilon-1}\right)\right)\exp\left(\int_t^\tau\left(g_u^{\pi^N}-r_u\right)du\right)d\tau\\ +\int_{\widehat{t}+T}^{\infty}\psi H_t^P\frac{\left(1+\varphi w_\tau^{\epsilon-1}\right)^\mu-1}{G_t\left(1+\varphi w_t^{\epsilon-1}\right)^\mu+1}\exp\left(\int_t^\tau\left(g_u^{\pi^N}-r_u\right)du\right)d\tau\end{array}\right).$$

Further, we know that $r_u=\rho+\theta g_u^C$ with $\theta\geq 1$. In addition $C_u=Y_u-X_u$, with X_u the aggregate spending on machines (initially negligible and later on a share of output bounded away from 1) and π_u^N initially grows like Y_u/N_u (and from then on will grow slower), therefore we have that $r_u-g_u^{\pi^N}>\rho$. Hence one can write:

$$\begin{aligned}&\left(\widehat{h}_t^A\right)^{1-\kappa}\\ &\leq\kappa\eta G_t^{\widetilde{\kappa}}\left(\int_t^{\widehat{t}+T}\mu\psi H_t^P\varphi w_\tau^{\epsilon-1}\exp\left(\int_t^\tau\left(g_u^{\pi^N}-r_u\right)du\right)d\tau+o\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)+o\left(e^{-\rho\left(T+\widehat{t}-t\right)}\right)\right)\end{aligned}$$

Since $r_u-g_u^{\pi^N}>\rho$, there exists a $\phi>0$, such that

$$\begin{aligned}\int_t^{\widehat{t}+T}\exp\left(\int_t^\tau\left(g_u^{\pi^N}-r_u\right)du\right)d\tau&\leq\int_t^{\widehat{t}+T}e^{-(\rho+\phi)(\tau-t)}d\tau\\ &\leq\frac{1}{\rho+\phi}\left(1-e^{-(\rho+\phi)\left(\widehat{t}+T-t\right)}\right).\end{aligned}$$

This allows us to rewrite:

$$\left(\widehat{h}_t^A\right)^{1-\kappa}\leq\kappa\eta G_t^{\widetilde{\kappa}}\left(\frac{\mu\psi H_t^P\varphi w_{\widehat{t}+T}^{\epsilon-1}}{\rho}+o\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)+o\left(e^{-\rho T}\right)\right)$$

Therefore, since T is large and $\varphi w_{\widehat{t}+T}^{\epsilon-1}$ is small, then $\widehat{h}_t^A$ must be small too. In fact, we get that $\widehat{h}_t^A = O\left(\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}\right) + o\left(e^{-\rho T}\right)$.

For any $t \in \left(0, \widehat{t}\right)$, we can then rewrite (31) as

$$\frac{\dot{\chi}_t}{\chi_t} = \gamma\psi H^P - \rho - (\theta\psi - \psi + 1)\, g_t^N + O\left(\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}\right) + o\left(e^{-\rho T}\right). \tag{60}$$

Using (36) we obtain

$$C_t = Y_t - X_t = \left(1 + O\left(G_t \varphi w_t^{(\varepsilon-1)}\right)\right) Y_t.$$

Next (5) and the corresponding equation for high-skill labor demand in production imply:

$$\frac{L^{NA}}{L^A} = \frac{(1-G_t)\left(1+\varphi w_t^{\epsilon-1}\right)^{-\mu-1}}{G_t} \text{ and } \frac{H^{P,NA}}{H^{P,A}} = \frac{(1-G_t)\left(1+\varphi w_t^{\epsilon-1}\right)^{-\mu}}{G_t}.$$

Using (3), we can then write

$$Y_t = N^{\frac{1}{\sigma-1}} L^\beta \left(H_t^P\right)^{1-\beta} \times$$

$$\left(G_t\left[1 + O\left(\varphi w^{\epsilon-1}\right) + \left(O\left(\varphi w_t^{(\varepsilon-1)}\right)\varphi^{\frac{1}{\varepsilon}}\frac{Y_t}{L}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon\beta}{\epsilon-1}\frac{\sigma-1}{\sigma}} \left(1 + O\left(\varphi w_t^{\epsilon-1}\right)\right) + 1 - G + O\left(\varphi w_t^{\epsilon-1}\right)\right)^{\frac{\sigma}{\sigma-1}}$$

Note that we have $w_t = O\left(Y_t/L\right)$ therefore $\varphi^{\frac{1}{\varepsilon}} Y_t/L = O\left(\varphi^{\frac{1}{\varepsilon}} w_t\right)$. Therefore

$$Y_t = \left(1 + O\left(\varphi w_t^{\epsilon-1}\right)\right) N^{\frac{1}{\sigma-1}} L^\beta \left(H_t^P\right)^{1-\beta}.$$

From this, using (8), one obtains that high-skill wages obey:

$$v_t = \left(1 + O\left(\varphi w_t^{\epsilon-1}\right)\right) \frac{\sigma-1}{\sigma} (1-\beta)\, N_t^{\frac{1}{\sigma-1}} L^\beta \left(H_t^P\right)^{-\beta},$$

while for low-skill wages, we get

$$w_t = \left(1 + O\left(\varphi w_t^{\epsilon-1}\right)\right) \frac{\sigma-1}{\sigma} \beta N_t^{\frac{1}{\sigma-1}} L^{\beta-1} \left(H_t^P\right)^{1-\beta}. \tag{61}$$

Therefore using the definition of χ_t, we obtain that

$$\chi_t = \left(1 + O\left(\varphi w_t^{(\varepsilon-1)}\right)\right) \sigma\psi L^{\beta(\theta-1)} \left(H_t^P\right)^{(1-\beta)\theta+\beta} N_t^{\frac{(1-\theta)\beta}{(\sigma-1)(1-\beta)}}.$$

Differentiating and plugging into (60) and using (38), we get (recalling (61) so that $d\ln\left(1 + O\left(\varphi w_t^{(\varepsilon-1)}\right)\right)/dt$ will be of order $O\left(\varphi w_t^{(\varepsilon-1)}\right)$ as well).

$$((1-\beta)\theta+\beta)\frac{\dot{H}_t^P}{H_t^P} = \gamma\psi H_t^P - \rho - \left(\frac{\theta-1}{\sigma-1}+1\right)\gamma\left(H - H_t^P\right) + O\left(\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}\right) + o\left(e^{-\rho T}\right), \tag{62}$$

we dropped terms in $\varphi w_t^{\epsilon-1}$ since there will negligible in front of $\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}$. The exact counterpart of this system admits a BGP with H_t^P constant, and as in the Romer (1990), there is no transitional dynamics. Therefore, here, we must have over the interval $\left(0, \widehat{t}\right)$

$$H_t^P = \frac{\left(\frac{\theta-1}{\sigma-1}+1\right)H + \frac{\rho}{\gamma}}{\psi + \frac{\theta-1}{\sigma-1}+1} + O\left(\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}\right) + o\left(e^{-\rho T}\right)$$

$$\text{and } g_t^N = \frac{\gamma H\psi - \rho}{\psi + \frac{\theta-1}{\sigma-1}+1} + O\left(\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}\right) + o\left(e^{-\rho T}\right)$$

which is positive under assumption (23). We then have that for N_t low, (20) can be solved as $G_t = G_0 \exp\left(-\frac{\gamma H\psi-\rho}{\psi+\frac{\theta-1}{\sigma-1}+1}t\right) + O\left(\left(\varphi w_{\widehat{t}+T}^{\epsilon-1}\right)^{\frac{1}{1-\kappa}}\right) + o\left(e^{-\rho T}\right)$. This characterizes the solution during Phase 1.

8.3.4 Transition from the first to the second phase

$\widetilde{\kappa} = 0$, Phase 1 cannot last forever as at some point, N_t and therefore w_t will become large. Since, the Poisson rate is $\eta\left(\widehat{h}_t^A\right)^{\kappa} = \Theta\left(\varphi w_t^{\varepsilon-1}\right)^{\frac{\kappa}{1-\kappa}}$, where $\Theta(z)$ denotes of the same order as z. This implies that G_t must start growing at a positive rate and that we enter the second phase.

When $\widetilde{\kappa} > 0$ (and $G_0 \neq 0$, otherwise automation is impossible), however, whether the Poisson rate of automation becomes negligible or not depends on a horse race between the drop in the share of automated products (and therefore the efficiency of the automation technology) and the rise in the low-skill wages (which, through horizontal innovation can become arbitrarily large). We look for a sufficient condition under which the Poisson rate will take off.

First assume that $G_t w_t^{\beta(\sigma-1)}$ does not tend towards 0. Then from (59) we obtain that:

$$\widehat{h}_t^A = \widehat{h}_t^A \left(G_t^{\widetilde{\kappa}-1}\right)^{\frac{1}{1-\kappa}} \Longrightarrow \eta G_t^{\widetilde{\kappa}} \left(\widehat{h}_t^A\right)^{\kappa} = \Theta\left(G_t^{\frac{\widetilde{\kappa}-\kappa}{1-\kappa}}\right) \tag{63}$$

Since $\widetilde{\kappa} \leq \kappa$, we obtain that the Poisson rate of automation diverges: a contradiction.

Assume now that $G_t w_t^{\beta(\sigma-1)}$ does tend towards 0. This ensures that $w_t = \Theta\left(N_t^{\frac{1}{\sigma-1}}\right)$. Moreover, $\frac{\pi_t^A - \pi_t^N}{G_t \pi_t^A + (1-G_t)\pi_t^N} = \Theta\left(w_t^{\beta(\sigma-1)}\right)$. Then using this in (59), we obtain

$$\widehat{h}_t^A = \Theta\left(G_t^{\widetilde{\kappa}} w_t^{\beta(\sigma-1)}\right)^{\frac{1}{1-\kappa}}.$$

Note that $\widehat{h}_t^A$ must remain bounded otherwise high-skill labor market clearing is violated. Therefore, we must have $G_t^{\widetilde{\kappa}} w_t^{\beta(\sigma-1)}$ bounded (which implies that $G_t w_t^{\beta(\sigma-1)}$ tends towards 0). Therefore the Poisson rate obeys:

$$\eta G_t^{\widetilde{\kappa}} \left(\widehat{h}_t^A\right)^{\kappa} = \Theta\left(G_t^{\frac{\widetilde{\kappa}}{1-\kappa}} N_t^{\frac{\beta\kappa}{1-\kappa}}\right)$$

Plugging this in (28) we get:

$$\dot{G}_t = \Theta\left(G_t^{\frac{\widetilde{\kappa}}{1-\kappa}} N_t^{\frac{\beta\kappa}{1-\kappa}}\right) - g_t^N G_t$$

To obtain that the share G_t is going towards 0, it must first be that $G_t^{\frac{\widetilde{\kappa}}{1-\kappa}} N_t^{\frac{\beta\kappa}{1-\kappa}}$ declines at the same rate or faster than G_t.

Consider first the case where, $G_t^{\frac{\widetilde{\kappa}}{1-\kappa}} N_t^{\frac{\beta\kappa}{1-\kappa}}$ and G_t are of the same order. In that case, we must have:

$$G_t = \Theta\left(N_t^{\frac{\beta\kappa}{1-\kappa-\widetilde{\kappa}}}\right)$$

This cannot go towards 0 if $1-\kappa-\widetilde{\kappa} > 0$. In addition, recall that this reasoning assumed that $G_t^{\widetilde{\kappa}} w_t^{\beta(\sigma-1)}$ remains bounded. We have

$$G_t^{\widetilde{\kappa}} w_t^{\beta(\sigma-1)} = \Theta\left(N_t^{\frac{\beta(1-\kappa)(1-\widetilde{\kappa})}{1-\kappa-\widetilde{\kappa}}}\right),$$

which is indeed declining if $1-\kappa-\widetilde{\kappa} < 0$.

Alternatively, if $G_t^{\frac{\widetilde{\kappa}}{1-\kappa}} N_t^{\frac{\beta\kappa}{1-\kappa}}$ goes towards 0 faster than G_t then G_t will be declining at the rate g_t^N, so that we have $G_t = \Theta\left(N_t^{-1}\right)$. This then implies

$$G_t^{\frac{\widetilde{\kappa}}{1-\kappa}} N_t^{\frac{\beta\kappa}{1-\kappa}} / G_t = \Theta\left(N_t^{\frac{\beta\kappa+1-\kappa-\widetilde{\kappa}}{1-\kappa}}\right).$$

As soon as $1-\kappa-\widetilde{\kappa} > 0$ then this cannot go towards 0.

Therefore $1-\kappa-\widetilde{\kappa} > 0$ is a sufficient condition which ensures that the Poisson rate of automation must take off.

8.3.5 Morishima elasticities of substitution

Given a production function which uses low-skill and high-skill labor, the Morishima elasticity of substitution between the two inputs is defined as

$$\varepsilon_{L,H}^M \equiv \frac{\partial}{\partial \ln v} \ln \frac{c_w}{c_v},$$

where c is the associated cost function. In our model we can similarly define an elasticity of substitution starting from the aggregate cost function:

$$c\left(v, w, p_x\right) = \frac{\sigma N^{\frac{1}{1-\sigma}} v^{1-\beta}}{\left(\sigma-1\right)\beta^\beta \left(1-\beta\right)^{1-\beta}} \left(G\left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^\mu + \left(1-G\right) w^{\beta(1-\sigma)}\right)^{\frac{1}{1-\sigma}}. \quad (64)$$

Note that this cost function simply extends equation (11) to the case where machines' price is given by p_x. Differentiating (64) with respect to high-skill and low-skill wages one gets:

$$\frac{c_w}{c_v} = \frac{\beta v \left(G\left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1} w^{-\varepsilon} + \left(1-G\right) w^{\beta(1-\sigma)-1}\right)}{\left(1-\beta\right)\left(G\left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^\mu + \left(1-G\right) w^{\beta(1-\sigma)}\right)}.$$

It is then direct to obtain the elasticity of substitution between low-skill and high-skill labor:

$$\varepsilon_{L,H}^M = 1.$$

With three factors, the elasticity of substitution need not be symmetric and indeed we obtain that the elasticity between high-skill and low-skill labor is given by:

$$\begin{aligned}
\varepsilon^M_{H,L} &\equiv \frac{\partial}{\partial \ln w} \ln \frac{c_v}{c_w} \\
&= 1 + \frac{\beta(\sigma-1)\varphi p_x^{1-\varepsilon} G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1}}{\left(G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu} + (1-G)\, w^{\beta(1-\sigma)}\right)} \\
&\quad + \frac{(1-\mu)\,(\varepsilon-1)\, G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1} w^{-\varepsilon}}{G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1} w^{-\varepsilon} + (1-G)\, w^{\beta(1-\sigma)-1}} \frac{\varphi p_x^{1-\varepsilon}}{\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}} \\
&> 1.
\end{aligned}$$

Note that when w is small with respect to p_x, then $\varepsilon^M_{H,L}$ is close to 1. This occurs when N is small (since $p_x = 1$), that is during the first phase. On the contrary when w is large (which is the case asymptotically), then as long as G is bounded above 0, we get that $\varepsilon^M_{H,L}$ is close to $1 + \beta(\sigma - 1)$. Therefore the elasticity of substitution between high-skill and low-skill labor will have increased between Phase 3 and Phase 1.

Similarly we can derive

$$\frac{c_{p_x}}{c_v} = \frac{\beta v G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1} p_x^{-\varepsilon}}{(1-\beta)\left(G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu} + (1-G)\, w^{\beta(1-\sigma)}\right)}.$$

This gives that $\varepsilon^M_{X,H} = 1$, while the elasticity of substitution between high-skill and machines obeys:

$$\begin{aligned}
\varepsilon^M_{H,X} &\equiv \frac{\partial}{\partial \ln p_x} \ln \frac{c_v}{c_{p_x}} \\
&= 1 + \beta(\sigma-1) \frac{\varphi p_x^{1-\varepsilon}}{\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}} \frac{(1-G)\, w^{\beta(1-\sigma)}}{G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu} + (1-G)\, w^{\beta(1-\sigma)}} + \frac{w^{1-\varepsilon}}{\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}} (\varepsilon - 1) \\
&> 1
\end{aligned}$$

From this it is easy to derive that, as long as G is bonded below, $\varepsilon^M_{H,X}$ is close to ϵ when w is small relative to p_x, while it is close to 1 when w is large. Therefore, as the transition unfolds the elasticity of substitution between high-skill and machines will have decreased and at some point will become lower than that between high-skill labor and low-skill labor.

Finally, we compute

$$\frac{c_w}{c_{p_x}} = \frac{G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1} w^{-\varepsilon} + (1-G)\, w^{\beta(1-\sigma)-1}}{G \left(\varphi p_x^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu-1} p_x^{-\varepsilon}},$$

from which we get that the Morishima elasticity between machines and low-skill labor is symmetric, so that

$$\varepsilon_{L,X}^{M}=\varepsilon_{X,L}^{M}=\varepsilon-(1-\mu)(\varepsilon-1)\frac{\varphi p_x^{1-\varepsilon}}{\varphi p_x^{1-\varepsilon}+w^{1-\varepsilon}}\frac{(1-G)\,w^{\beta(1-\sigma)-1}}{G\left(\varphi p_x^{1-\varepsilon}+w^{1-\varepsilon}\right)^{\mu-1}w^{-\varepsilon}+(1-G)\,w^{\beta(1-\sigma)-1}}.$$

Looking at the limits (under the assumption that G is bonded below), we obtain $\varepsilon_{X,L}^{M}|_{w=0}=\varepsilon$ and $\varepsilon_{X,L}^{M}|_{w=\infty}=1+\beta(\sigma-1)$. Therefore the elasticity of substitution between machines and low-skill labor (and that between low-skill labor and machines) will have decreased during the second phase.

8.3.6 Comparative statics

In this section, we prove Proposition 4. The proposition is established when the steady state is unique but it extends to the case of the steady states with the highest and lowest growth rates when there is multiplicity. Recall that the steady state is characterized as the solution to an equation $f\left(g^{N*}\right)=1$ through (56), where G^{*}, $\widehat{h}^{A*}$ and H^{P*} can all be written as functions of g^{N*} and parameters. Moreover, when there is a single steady state (as well as for the steady states with the highest and the lowest growth rates in case of multiplicity), f must be increasing in the neighborhood of g^{N*}.

Comparative static with respect to γ. (52) implies that $\widehat{h}^{A*}$ is inversely proportional to γ (for given g^{N*}). Formally, we have:

$$\frac{\partial\widehat{h}^{A*}}{\partial\gamma}=-\frac{\widehat{h}^{A*}}{\gamma}. \tag{65}$$

Differentiating (53) and using (65) leads to:

$$\frac{\partial G^{*}}{\partial\gamma}=\frac{-\kappa g^{N*}G^{*}}{\gamma\left(\eta\left(G^{*}\right)^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}+(1-\widetilde{\kappa})\,g^{N*}\right)}, \tag{66}$$

so that for a given g^{N*}, G^{*} is also decreasing in γ. Using (54), (65) and (66), we get:

$$\frac{\partial H^{P*}}{\partial\gamma}=\frac{1}{\gamma}\left(\begin{array}{c}\frac{g^{N*}}{\gamma}+(1-G^{*})(1-\kappa)\,\widehat{h}^{A*}\\+\frac{(1-\widetilde{\kappa})\kappa\widehat{h}^{A*}G^{*}\left(g^{N*}\right)^{2}}{\left(\eta(G^{*})^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}\right)\left(\eta(G^{*})^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}+(1-\widetilde{\kappa})g^{N*}\right)}\end{array}\right)>0$$

so that H^{P*} is increasing in γ. Note that f, defined in (56), can be rewritten as

$$f\left(g^{N*}\right)=\frac{1-\kappa}{\kappa}\frac{1}{\psi H^{P*}}\left(\frac{\left(G^{*}\right)^{1-\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{1-\kappa}\left(\gamma\widehat{h}^{A*}\right)}{\kappa\eta}+G^{*}\widehat{h}^{A*}\right),$$

which shows that f is decreasing in γ for a given g^{N*} (H^{P*} is increasing, G^{*} and $\widehat{h}^{A*}$ are decreasing, and $\gamma\widehat{h}^{A*}$ is constant). Since f is increasing in g^{N*} at the equilibrium value, (56) implies that g^{N*} increases in γ. Moreover, following (53), G^{*} is decreasing in g^{N*} and in γ for a given g^{N*}, this implies that G^{*} decreases in γ (taking into account changes in g^{N*}).

Comparative static with respect to η. For given g^{N*}, (52) implies that $\widehat{h}^{A*}$ does not depend on η. Differentiating (53), we get:

$$\frac{\partial\ln G^{*}}{\partial\ln\eta}=\frac{g^{N*}}{\eta\left(G^{*}\right)^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}+\left(1-\widetilde{\kappa}\right)g^{N*}},\tag{67}$$

so for given g^{N*}, G^{*} increases in η. (54) implies then that

$$\frac{\partial\ln H^{P*}}{\partial\ln\eta}=\frac{G^{*}\widehat{h}^{A*}}{H^{P*}}\frac{g^{N*}}{\eta\left(G^{*}\right)^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}+\left(1-\widetilde{\kappa}\right)g^{N*}}.$$

Using this equation together with (67) and (56), we obtain:

$$\frac{\partial\ln f}{\partial\ln\eta}=\left\{\begin{array}{c}\frac{g^{N*}}{\eta(G^{*})^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}+(1-\widetilde{\kappa})g^{N*}}\left(1-\frac{G^{*}\widehat{h}^{A*}}{H^{P*}}-\widetilde{\kappa}\frac{\frac{1}{\kappa\eta(G^{*})^{\widetilde{\kappa}}}\left(\widehat{h}^{A*}\right)^{1-\kappa}}{\frac{1}{\kappa\eta(G^{*})^{\widetilde{\kappa}}}\left(\widehat{h}^{A*}\right)^{1-\kappa}+\frac{1}{\gamma}}\right)\\-\frac{\frac{1}{\kappa\eta(G^{*})^{\widetilde{\kappa}}}\left(\widehat{h}^{A*}\right)^{1-\kappa}}{\frac{1}{\kappa\eta(G^{*})^{\widetilde{\kappa}}}\left(\widehat{h}^{A*}\right)^{1-\kappa}+\frac{1}{\gamma}}\end{array}\right\}.$$

Using (52), we can rewrite this as:

$$\frac{\partial\ln f}{\partial\ln\eta}=\left\{\begin{array}{c}-\frac{1}{\eta(G^{*})^{\widetilde{\kappa}}\left(\widehat{h}^{A*}\right)^{\kappa}+(1-\widetilde{\kappa})g^{N*}}\\\times\left(\frac{g^{N*}G^{*}\widehat{h}^{A*}}{H^{P*}}+\frac{\rho+((\theta-1)\psi+\kappa)g^{N*}}{\gamma(1-\kappa)\left(\frac{1}{\eta_{L}\kappa G_{L}^{\widetilde{\kappa}}}\left(\widehat{h_{L\infty}}\right)^{1-\kappa}+\frac{1}{\gamma}\right)}\right)\end{array}\right\},$$

so that f is decreasing in η. This implies that g^{N*} must be increasing in η. Since $\widehat{h}^{A*}$

only depends on η through g^{N*}, we also get that $\widehat{h}^{A*}$ increases in η. The impact on G^* is ambiguous because G^* increases in η for a given g^{N*} but it is also decreasing in g^{N*}.

8.4 Simulation technique

In the following we describe the simulation techniques employed for the baseline model presented in 2. The approach for the extensions follow straightforwardly. Let $\mathbf{x}_t \equiv (n_t, G_t, \hat{h}_t^A, \chi_t, \omega_t)$ and note that equation (39) defines ω implicitly. We can therefore write equations (27), (28), (30) and (31) as a system of autonomous differential equations $(\dot{n}_t, \dot{G}_t, \dot{\hat{h}}_t^A, \dot{\chi}_t) = F(\mathbf{x}_t)$ with initial conditions on state variables as (n_0, G_0) and an auxiliary equation of $\omega_t = \vartheta(\mathbf{x}_t)$.

For the numerical solution, we specify a (small) time period of $dt > 0$ and a (large) number of time periods T. Using this we approximate the four differential equations by $(T-1) \times 4$ errors as:

$$s_t = (\frac{n_{t+1} - n_t}{dt}, \frac{G_{t+1} - G_t}{dt}, \frac{\hat{h}_{t+1}^A - \hat{h}_t^A}{dt}, \frac{\chi_{t+1} - \chi_t}{dt}) - F((\mathbf{x}_t + \mathbf{x}_{t+1})/2),\ t = \{1, ...T-1\}$$

with T corresponding errors for ω_t :

$$s_t^\omega = \omega_t - \vartheta(\mathbf{x}_t),\ t = \{1, ..., T\}.$$

As shown in Appendix 8.3.1 for a set of parameter values, the system admits an asymptotic steady state. We assume that the system has reached this asymptotic steady state by time T and restrict $\hat{h}_T^A$ and χ_T accordingly. Together with the initial conditions ($n_1 = n^{start}$ and $G_1 = G^{start}$) this leads to a vector of errors:

$$s_T \equiv (n_1 - n^{start}, G_1 - G^{start}, \hat{h}_T^A - \hat{h}^{A*}, \chi_T - \chi^*)'.$$

Letting $\mathbf{x} = \{\mathbf{x}_t\}_{t=1}^T$, we then stack errors to get a vector, $S(\mathbf{x})$, of length $5T$ and solve the following problem:

$$\hat{\mathbf{x}} = argmin_{\mathbf{x}} S(\mathbf{x})' W S(\mathbf{x}),$$

for a $5T \times 5T$ diagonal weighting matrix, W, and the $T \times 5$ matrix $\mathbf{x}$. For $dt \to 0$ and $T \to \infty$ $S(\mathbf{x})'WS(\mathbf{x}) \to 0$. For the simulations we set $dt = 2$ and $T = 2000$. We accept the solution when $S(\hat{\mathbf{x}})'WS(\hat{x}) < 10^{(-7)}$, but the value is typically less than

$10^{(-20)}$. The choice of weighting matrix matters somewhat for the speed of convergence, but is inconsequential for the final result. With the solution $\{\hat{\mathbf{x}}_t, \hat{\omega}_t\}_{t=1}^T$ in hand it is straightforward to find remaining predicted values.

8.5 Social planner problem

This section presents the solution to the social planner problem. After having set-up the problem, we derive the optimal allocation, emphasizing in particular the different inefficiencies in our competitive equilibrium. Then, we show the optimal allocation for our baseline parameters. Finally, we derive how the optimal allocation can be decentralized.

8.5.1 Characterizing the optimal allocation

We introduce the following notations: N_t^A (respectively N_t^N) denotes the mass of automated (respectively non-automated) firms, L_t^A (respectively L_t^N) is the mass of low-skill workers hired in automated (respectively non-automated) firms, and $H_t^{P,A}$ (respectively $H_t^{P,N}$) is the mass of high-skill workers hired in production in automated (respectively non-automated) firms. The social planner problem can then be written as (we write the Lagrange multipliers next to each constraint):

$$\max \int_0^\infty e^{-\rho t} \frac{C_t^{1-\theta}}{1-\theta}$$

such that

$$\widetilde{\lambda}_t : C_t + X_t = F\left(L_t^A, H_t^{P,A}, X_t, L_t^N, H_t^{P,N}, N_t^A, N_t^N\right),$$

with

$$F \equiv \left(\begin{array}{c} \left(N_t^A\right)^{\frac{1}{\sigma}} \left(\left(\widetilde{\varphi_L X_t}^{\frac{\varepsilon-1}{\varepsilon}} + \left(L_t^A\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}\beta} \left(H_t^{P,A}\right)^{1-\beta}\right)^{\frac{\sigma-1}{\sigma}} \\ + \left(N_t^N\right)^{\frac{1}{\sigma}} \left(\left(L_t^N\right)^{\beta} \left(H_t^{P,N}\right)^{1-\beta}\right)^{\frac{\sigma-1}{\sigma}} \end{array} \right)^{\frac{\sigma}{\sigma-1}},$$

$$\widetilde{w}_t : L_t^A + L_t^N = L,$$

$$\widetilde{v}_t : H_t^{P,A} + H_t^{P,N} + H_t^A + H_t^D = H,$$

$$\widetilde{\zeta}_t : \dot{N}_t^N = \gamma\left(N_t^A + N_t^N\right) H_t^D - \eta\left(N_t^A\right)^{\widetilde{\kappa}} \left(N_t^N + N_t^A\right)^{\kappa-\widetilde{\kappa}} \left(H_t^A\right)^{\kappa} \left(N_t^N\right)^{1-\kappa},$$

$$\widetilde{\xi}_t : \dot{N}_t^A = \eta\left(N_t^A\right)^{\widetilde{\kappa}} \left(N_t^N + N_t^A\right)^{\kappa-\widetilde{\kappa}} \left(H_t^A\right)^{\kappa} \left(N_t^N\right)^{1-\kappa},$$

$$H_t^D \geq 0.$$

The first order condition with respect to C_t gives

$$C_t^{-\theta} = \widetilde{\lambda}_t$$

To denote the ratio of the Lagrange parameter of each constraint with respect to $\widetilde{\lambda}_t$ (that is the shadow value expressed in units of final good at time t), we remove the tilde (hence $w_t \equiv \widetilde{w}_t / \widetilde{\lambda}_t$ is the shadow wage of low-skill workers).

The first order conditions with respect to X_t implies that

$$\frac{\partial F}{\partial X_t} = 1, \tag{68}$$

so that the shadow price of a machine must be equal to 1. First order conditions with respect to L_t^A, L_t^N, $H_t^{P,A}$, $H_t^{P,N}$ lead to

$$w_t = \frac{\partial F}{\partial L_t^A} = \frac{\partial F}{\partial L_t^N} \text{ and } v_t = \frac{\partial F}{\partial H_t^{P,A}} = \frac{\partial F}{\partial H_t^{P,N}}, \tag{69}$$

so that labor inputs are paid their marginal product in aggregate production. This is not the case in the competitive equilibrium, where labor inputs are paid their marginal product in the production of intermediates, while intermediates themselves are priced with a mark-up as they are provided by a monopolist. It is easy to show that for a given H_t^P, the optimal provision of machines and allocation of high-skill and low-skill workers across firms can be obtained if the purchase of all intermediate inputs is subsidized by at rate $1/\sigma$ (a lump-sum tax finances the subsidy).

The first-order conditions with respect to N_t^N and N_t^A are given by:

$$\rho \widetilde{\zeta}_t - \dot{\widetilde{\zeta}}_t = \begin{array}{c} \widetilde{\lambda}_t \frac{\partial F}{\partial N_t^N} + \widetilde{\zeta}_t \gamma H_t^D + \left(\widetilde{\xi}_t - \widetilde{\zeta}_t\right) \eta \left(H_t^A\right)^{\kappa} \left(N_t^N\right)^{-\kappa} \\ \times \left(N_t^A\right)^{\widetilde{\kappa}} \left(\left(1-\widetilde{\kappa}\right) N_t^N + \left(1-\kappa\right) N_t^A\right) \left(N_t^N + N_t^A\right)^{\kappa - \widetilde{\kappa} - 1} \end{array}, \tag{70}$$

$$\rho \widetilde{\xi}_t - \dot{\widetilde{\xi}}_t = \begin{array}{c} \widetilde{\lambda}_t \frac{\partial F}{\partial N_t^A} + \widetilde{\zeta}_t \gamma H_t^D + \left(\widetilde{\xi}_t - \widetilde{\zeta}_t\right) \eta \left(H_t^A\right)^{\kappa} \\ \times \left(N_t^N\right)^{1-\kappa} \left(N_t^A\right)^{\widetilde{\kappa}-1} \left(\widetilde{\kappa} N_t^N + \kappa N_t^A\right) \left(N_t^N + N_t^A\right)^{\kappa - \widetilde{\kappa} - 1} \end{array}. \tag{71}$$

Interestingly, $\frac{\partial F_t}{\partial N_t^N}$ and $\frac{\partial F}{\partial N_t^A}$ correspond to the profits realized by a non-automated and an automated firm respectively in the equilibrium once the subsidy to the use of intermediates is implemented. Therefore we denote

$$\pi_t^N = \frac{\partial F_t}{\partial N_t^N} \text{ and } \pi_t^A = \frac{\partial F_t}{\partial N_t^A}$$

Further the (shadow) interest rate is given by $r_t = \rho + \theta \frac{\dot{C}_t}{C_t} = \rho - \frac{\dot{\lambda}_t}{\lambda_t}$. Using that $H_t^A = (1 - G_t) N_t h_t^A$, we can rewrite (70) and (71) as:

$$r_t \zeta_t = \pi_t^N + \zeta_t g_t^N + (\xi_t - \zeta_t) \eta (G_t)^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa} ((1 - \widetilde{\kappa})(1 - G_t) + (1 - \kappa) G_t) + \dot{\zeta}_t, \quad (72)$$

$$r_t \xi_t = \pi_t^A + \zeta_t g_t^N + (\xi_t - \zeta_t) \eta (G_t)^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa} (1 - G_t) \left(\widetilde{\kappa} \frac{(1 - G_t)}{G_t} + \kappa\right) + \dot{\xi}_t. \quad (73)$$

These expressions parallel equations (17) and (18) in the paper. The rental social value of a non-automated firm ($r_t \zeta_t$) consists of the current value of one intermediate (which equals the profits when the optimal subsidy to the use of intermediates inputs is in place), its positive impact on the horizontal innovation technology (the productivity of which is γN_t), its positive impact on the automation technology (which results from the direct externality embedded in the automation technology from the number of firms diminished by the additional externality coming from the share of automated products), the expected increase in its value if it becomes automated minus the cost of the resources required (the difference between these two terms is positive since the automation technology is concave) and the change in its value. The rental social value of an automated firms ($r_t \xi_t$) is the sum of the profits, its impact on horizontal innovation (through the same externality as non-automated firm), its impact on the automation technology (which results from two externalities as both the number of firms and the share of automated products improve the automation technology), and the change in its value.

The first order condition with respect to H_t^D gives (together with the condition that $H_t^D \geq 0$):

$$v_t \geq \zeta_t \gamma N_t, \quad (74)$$

with equality when $H_t^D > 0$. This equation is the counterpart of (21) in the equilibrium case, it stipulates that when horizontal innovation takes place the social value of a non-automated intermediate equals the cost of creating one. The first-order condition with respect to H_t^A gives:

$$v_t = (\xi_t - \zeta_t) \kappa \eta (G_t)^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa - 1}. \quad (75)$$

This equation is the counterpart of (19) in the equilibrium case. Everything else given, $\xi_t - \zeta_t$ increases with $\pi_t^A - \pi_t^N$, which increases with w_t, therefore this equation shows that automation increases with low-skill wages (everything else given), just as in the equilibrium case.

8.5.2 System of differential equations and steady state

After having introduced the same variables as in the equilibrium case, one can follow the same steps and derive a system of differential equation in $\left(n_t, G_t, \widehat{h}_t^A, \chi_t\right)$ which characterizes the solution (when there is positive growth). Equations (27) and (28) still hold, while equations (30) and (31) are replaced with

$$\dot{\widehat{h}}_t^A = \begin{array}{c} \frac{\gamma \widehat{h}_t^A}{1-\kappa}\left(\omega_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{-\mu}\frac{\widehat{\pi}_t^A}{\widehat{v}_t}+\frac{1-\kappa+(\kappa-\widetilde{\kappa})(1-G_t)}{\kappa}\widehat{h}_t^A\right) \\ -\frac{\eta\kappa G_t^{\widetilde{\kappa}}}{1-\kappa}\left(\widehat{h}_t^A\right)^{\kappa}\left(1-\omega_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{-\mu}\right)\frac{\widehat{\pi}_t^A}{\widehat{v}_t}+\eta G_t^{\widetilde{\kappa}}\left(\widehat{h}_t^A\right)^{\kappa+1}+\frac{1-\widetilde{\kappa}}{1-\kappa}g_t^N\widehat{h}_t^A \end{array}, \tag{76}$$

$$\dot{\chi}_t = \chi_t\left(\gamma\omega_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{-\mu}\frac{\widehat{\pi}_t^A}{\widehat{v}_t}+\gamma\frac{1-\kappa+(\kappa-\widetilde{\kappa})(1-G_t)}{\kappa}\widehat{h}_t^A-\rho-(\theta-1)\psi g_t^N\right).$$

g_t^N is still given by (38), $\frac{\widehat{\pi}_t^A}{\widehat{v}_t}$, H_t^P and ω_t are now given by

$$\frac{\widehat{\pi_t^A}}{\widehat{v}_t} = \frac{\psi\left(\varphi_L+\omega_t^{\frac{1}{\mu}}\right)^{\mu}H_t^P}{G_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}+(1-G_t)\,\omega_t},$$

$$H_t^P = \frac{(1-\beta)^{\frac{1}{\theta}}\beta^{\frac{\beta}{1-\beta}\left(\frac{1}{\theta}-1\right)}\chi_t^{\frac{1}{\theta}}\left(G_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}+(1-G_t)\,\omega_t\right)^{\psi\left(\frac{1}{\theta}-1\right)+1}}{G_t\left((1-\beta)\,\varphi+\omega_t^{\frac{1}{\mu}}\right)\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu-1}+(1-G_t)\,\omega_t},$$

$$\omega_t = n_t\left(\begin{array}{c}\beta^{\frac{1}{1-\beta}}\frac{H_t^P}{L}\left(G_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu-1}\omega_t^{\frac{1-\mu}{\mu}}+(1-G_t)\right) \\ \times\left(G_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}+(1-G_t)\,\omega_t\right)^{\psi-1}\end{array}\right)^{\frac{\beta(1-\sigma)}{1+\beta(\sigma-1)}},$$

which replace (35), (37) and (39).

One can then solve for a steady state of this system with $G^* > 0$ (and $\left(g^N\right)^* > 0$ so that $n^* = 0$). (53) and (54) still apply, but (52) is replaced with

$$\widehat{h}^{A*} = \frac{\kappa}{\gamma} \frac{\rho + (\theta - 1)\,\psi g^{N*}}{1 - \kappa + (1 - G^*)\,(\kappa - \widetilde{\kappa})}, \tag{77}$$

and (56) with

$$f^{sp}\left(g^{N*}\right) \equiv \frac{\rho + (\theta - 1)\,\psi g^{N*}}{\psi H^{P*}} \left(\frac{\left(\widehat{h}^{A*}\right)^{1-\kappa}}{\eta\kappa\,(G^*)^{\widetilde{\kappa}-1}} + \frac{1}{\gamma} \right),$$

which is obtained by fixing $\dot{\widehat{h}}_t^{A} = 0$ in (76) using (53) and (77). For g^{N*} large enough (but finite—and, in particular smaller than γH), H^{P*} is arbitrarily small, while for the same value G^* and $\widehat{h}^{A*}$ are bounded below and above. As before, this establishes that for g^{N*} large enough $f^{sp}\left(g^{N*}\right) > 1$. Furthermore $f^{sp}(0) = f(0)$, therefore condition 23 is also a sufficient condition for the existence of a steady state with positive growth and $G^* > 0$ for the system of differential equations.

8.5.3 Decentralizing the optimal allocation

We have already seen that the "static" optimal allocation given H_t^P is identical to the equilibrium allocation once a subsidy to the use of intermediates $1/\sigma$ is in place. The "dynamic" part of the problem consists of the allocation of high-skill workers across the two types of innovation and production. Therefore, we postulate that a social planner can decentralize the optimal allocation using the subsidy to the use of intermediate inputs and subsidies (or taxes) for high-skill workers hired in automation $\left(s_t^A\right)$ and in horizontal innovation $\left(s_t^H\right)$. Let us consider such an equilibrium and introduce the notations $\Omega_t^A \equiv 1 - s_t^A$ and Ω_t^H similarly defined. In this situation, the law of motion for the private value of an automated firm, V_t^A, is still given by (17), for a non-automated firm it obeys:

$$r_t V_t^N = \pi_t^N - \Omega_t^A v_t h_t + \eta\,(G_t)^{\widetilde{\kappa}}\,N_t^{\kappa}\left(h_t^A\right)^{\kappa}\left(V_t^A - V_t^N\right) + \dot{V}_t^N, \tag{78}$$

instead of (18), the first-order condition for automation is given by:

$$\kappa\eta\,(G_t)^{\widetilde{\kappa}}\,N_t^{\kappa}\left(h_t^A\right)^{\kappa-1}\left(V_t^A - V_t^L\right) = \Omega_t^A v_t, \tag{79}$$

instead of (19), while the free entry condition, when $g_t^N > 0$, is given by

$$\gamma N_t V_t^N = \Omega_t^H v_t, \tag{80}$$

instead of (21). For Ω_t^A and Ω_t^H to decentralize the optimal allocation it must be that these 4 equations hold together with (72), (73), (74) and (75).

Using (74) and (80), we then get that Ω_t^H must satisfy

$$\Omega_t^H \zeta_t = V_t^N, \tag{81}$$

similarly, using (75) and (79), we get

$$\Omega_t^A (\xi_t - \zeta_t) = V_t^A - V_t^L. \tag{82}$$

Plugging (81) and (82) in (78), we get that

$$r_t \zeta_t = \frac{\pi_t^N}{\Omega_t^H} - \frac{\Omega_t^A}{\Omega_t^H} v_t h_t + \eta (G_t)^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa} \frac{\Omega_t^A}{\Omega_t^H} (\xi_t - \zeta_t) + \frac{\dot{\Omega}_t^H}{\Omega_t^H} \zeta_t + \dot{\zeta}_t. \tag{83}$$

Similarly, using (82) and the difference between (17) and (78) gives:

$$r_t (\xi_t - \zeta_t) = \frac{\pi_t^A - \pi_t^N}{\Omega_t^A} + v_t h_t - \eta (G_t)^{\widetilde{\kappa}} N_t^{\kappa} \left(h_t^A\right)^{\kappa} (\xi_t - \zeta_t) + \frac{\dot{\Omega}_t^A}{\Omega_t^A} (\xi_t - \zeta_t) + \dot{\xi}_t - \dot{\zeta}_t. \tag{84}$$

Combining (83) with (72), using (75) and (74) and the definition of Ω_t^A and Ω_t^H, we get:

$$\dot{s}_t^H = \begin{array}{c} \frac{\gamma \widehat{\pi}_t^N}{\widehat{v}_t} s_t^H - \left(1 - s_t^H\right) g_t^N \\ + \frac{\gamma \widehat{h}_t^A}{\kappa} \left(\left(1 - s_t^A\right)(1 - \kappa) + \left(1 - s_t^H\right) \left(\widetilde{\kappa}(1 - G_t) + \kappa G_t - 1\right) \right) \end{array}. \tag{85}$$

Similarly combining (84) with the difference between (73) and (72) and using (74) gives:

$$\dot{s}_t^A \frac{\left(\widehat{h}_t^A\right)^{1-\kappa}}{\eta G_t^{\widetilde{\kappa}}} = \kappa \frac{\widehat{\pi}_t^A - \widehat{\pi}_t^N}{\widehat{v}_t} s_t^A - \widetilde{\kappa} \left(1 - s_t^A\right) \widehat{h}_t^A \frac{1 - G_t}{G_t}. \tag{86}$$

Therefore, in steady state, we have

$$s_\infty^A = \frac{\widetilde{\kappa} \widehat{h}_\infty^A (1 - G_\infty)}{\kappa \psi H_\infty^P + \widetilde{\kappa} \widehat{h}_\infty^A (1 - G_\infty)} \geq 0.$$

Note from (86) that the share of automated products, s_t^A, must always be non-negative, otherwise it cannot converge to a positive value, therefore $s_t^A \geq 0$ everywhere (and in fact > 0 if $\widetilde{\kappa} \neq 0$). Furthermore, if $\widetilde{\kappa} = 0$, $s_t^A = 0$ everywhere, the only externality in automation comes from the total number of products, therefore the equilibrium features the optimal amount of automation investment (when the monopoly distortion is corrected and the optimal subsidy to horizontal innovation is implemented).

(85) gives the steady state value of the subsidy to horizontal innovation as:

$$s_\infty^H = 1 - \frac{\gamma \widehat{h}_\infty^A (1-\kappa)\left(1 - s_\infty^A\right)}{\kappa g_\infty^N + \gamma \widehat{h}_\infty^A (1 - \widetilde{\kappa}(1 - G_\infty) - \kappa G_\infty)}.$$

In addition, knowing that $s_t^A \geqslant 0$, imposes that $s_t^H > 0$—as $s_t^H < 0$ would lead to $\dot{s}_t^H < 0$.

8.5.4 Transitional dynamics for the social planner case

Figure 20 plots the transitional dynamics for the optimal allocation in our baseline case (which features $\widetilde{\kappa} = 0$) and in the case where $\widetilde{\kappa} = 0$ analyzed in Figure 4. As shown in Panel A and C, the economy also goes through three phases as a higher (shadow) low-skill wage leads to more automation over time and a transition from a small share to a high share of automated products. Relative to Figure 2.A and Figure 4.A, the overall dynamics look quite similar but the growth rates are higher in the social planner case, and the transition to phase 2 now happens roughly at the same time with and without the automation externality, while in the equilibrium it is considerably delayed in the presence of the externality (as, effectively, the productivity of the automation technology is initially very low). In both cases, the social planner maintains a positive subsidy to horizontal innovation. When $\widetilde{\kappa} = 0$ (without the automation externality), the subsidy to automation is 0, while when $\widetilde{\kappa} > 0$ there is a positive subsidy to automation, which is the largest in Phase 1. This subsidy explains why Phase 2 now starts at around the same time.

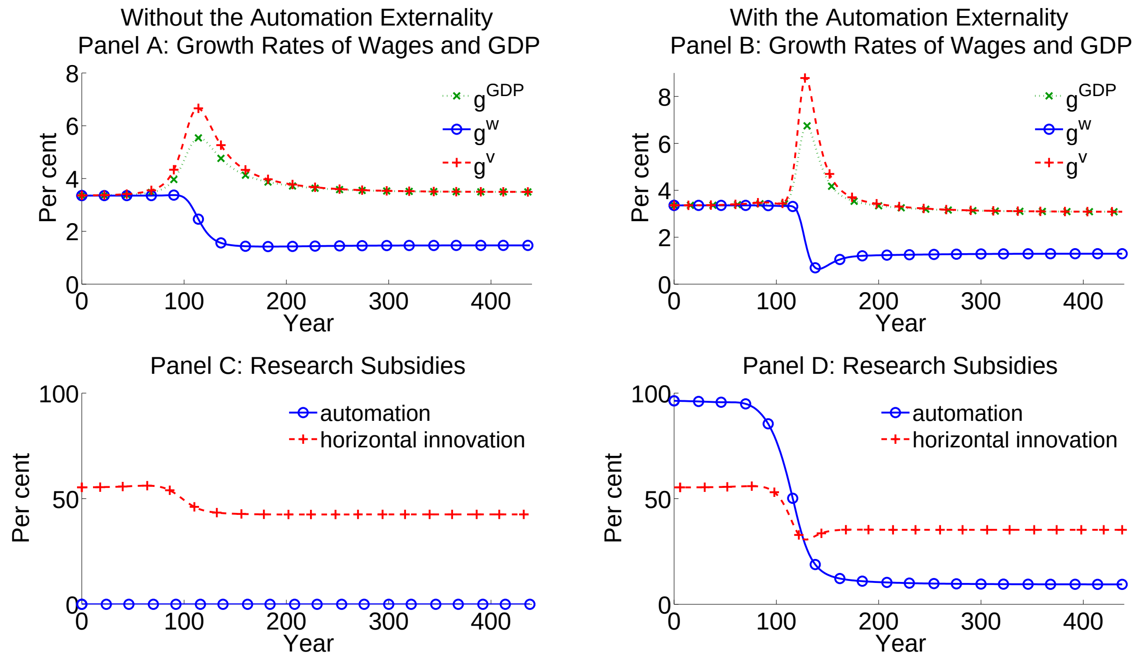

Figure 20: Transitional Dynamics in the Social Planner Case. Panel A and B, baseline case. Panel C and D, with $\tilde{\kappa} = 0.5$

8.6 Alternative model with automation at the entry-stage

To highlight that the evolution of the economy through three phases does not depend on our assumption that new products are born non-automated, we present in this section a model where, instead, we assume that automation can only take place at the entry stage. That is, when a new firm is born, it can hire h_t^A workers to automate it, in which case it is successful with probability $\min\left(\eta\left(N_t h_t^A\right)^{\kappa}, 1\right)$ (we abstract from the automation externality for simplicity). Ex-ante a firm does not know whether it will succeed or not, therefore, the free-entry condition can now be written as

$$v_t \geq \gamma N_t \overline{V_t},$$

where

$$\overline{V_t} = \min\left(\eta\left(N_t h_t^A\right)^{\kappa}, 1\right) V_t^A + \left(1 - \min\left(\eta\left(N_t h_t^A\right)^{\kappa}, 1\right)\right) V_t^N - v_t h_t^A.$$

is the expected value of a new firm. Since we used similar functional forms we have that h_t^A obeys (19) unless $\kappa \eta^{\frac{1}{\kappa}} N_t \left(V_t^A - V_t^L\right) > v_t$, in which case $N_t h_t^A = \eta^{-\frac{1}{\kappa}}$. Afterward a firm never becomes automated so that the law of motion for the value of an automated and a non-automated firms both follow (17). In addition, the law of motion for G_t is now given by

$$\dot{G}_t = g_t^N \left(\eta \left(N_t h_t^A\right)^{\kappa} - G_t\right).$$

The resolution of the model follows the same steps as in the baseline case, and under the appropriate condition on the discount rate, there exists an asymptotic steady state with $g_t^N > 0$.

An important difference is that G^* may be equal to 1 since all new products may choose to be automated in steady state. In fact, one can derive that $\widehat{h}^{A*} = \min\left(\eta^{-\frac{1}{\kappa}}, \frac{\kappa}{1-\kappa}\frac{1}{\gamma}\right)$. Therefore $G^* < 1$, if and only if $\eta\left(\frac{\kappa}{1-\kappa}\frac{1}{\gamma}\right)^{\kappa} < 1$. When $G^* < 1$, we will have that $G_\infty = G^* < 1$, so that, following Proposition 2,

$$g_\infty^w = \frac{1}{1+\beta(\sigma-1)} g_\infty^v.$$

On the contrary, if $G^* = 1$, then $G_\infty = 1$, and following Proposition 2, we get that

$$g_\infty^w = \frac{g_\infty^v}{\varepsilon}.$$

Figure 21 draws the transitional dynamics for the same parameters as in the baseline case (even though the automation technology parameters have a different meaning here). These parameters satisfy $\eta\left(\frac{\kappa}{1-\kappa}\frac{1}{\gamma}\right)^{\kappa} < 1$, and the figure shows that the economy goes through three phases as in our baseline model.

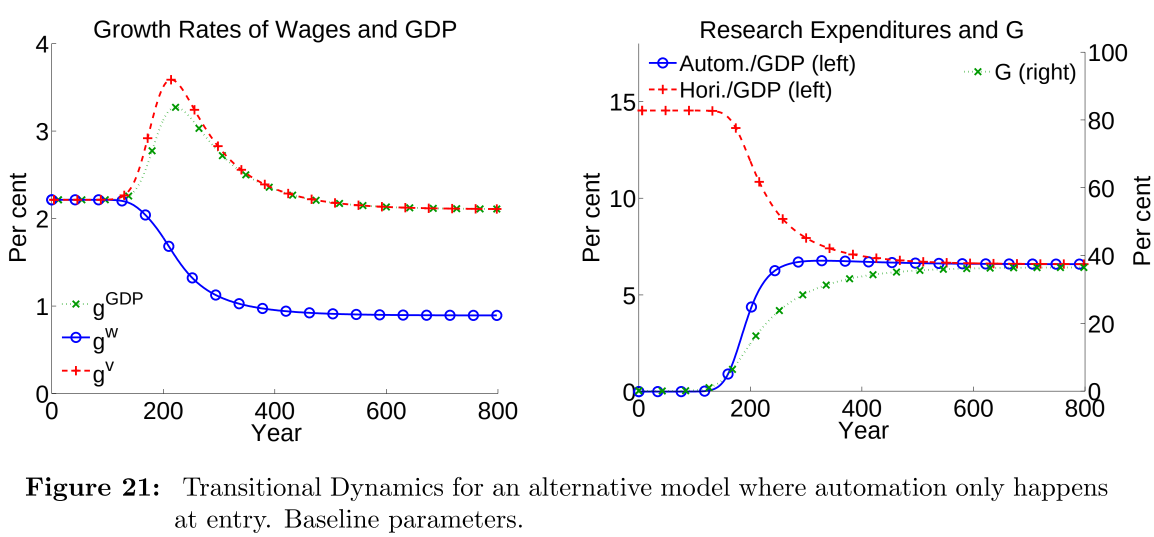

Figure 21: Transitional Dynamics for an alternative model where automation only happens at entry. Baseline parameters.

8.7 Supply response in the skill distribution

The supply of low-skill and high-skill labor are now endogenous. This does not affect (11) which still holds. (10) also holds with L_t replacing L and knowing that H_t^P obeys (16) but with H_t instead of H in the right-hand side. Because workers are ordered such that a worker with a higher index j supplies relatively more high-skill labor, then at all point in times there exists a threshold $\bar{j}_t$ such that workers $j \in (0, \bar{j}_t)$ supply low-skill labor and workers $j \in (\bar{j}_t, 1)$ supply high-skill labor. As a result, we get that the total mass of low-skill labor is:

$$L_t = l\overline{H}\bar{j}_t, \tag{87}$$

and the mass of high-skill labor is

$$H_t = \overline{H}\left(1 - \bar{j}_t^{\frac{1+q}{q}}\right) \leq \overline{H}. \tag{88}$$

The cut-off $\bar{j}_t$ obeys $l\overline{H}w_t = \Gamma\left(\bar{j}_t\right) v_t$, that is

$$\bar{j}_t = \left(\frac{q}{1+q}\frac{lw_t}{v_t}\right)^q. \tag{89}$$

$\bar{j}_t$ decreases as the skill premium increases and q measures the elasticity of $\bar{j}_t$ with respect to the skill premium.

8.7.1 Asymptotic growth rates

We consider processes $\left(N_t, G_t, H_t^P\right)$ such that g_t^N, G_t and H_t^P admit strictly positive limits. Plugging (89) and (87) in (10), we get:

$$\frac{v_t}{w_t} = l\left(\frac{1-\beta}{\beta}\frac{\overline{H}}{H_t^P}\left(\frac{q}{1+q}\right)^q \frac{G_t + (1-G_t)\left(1+\varphi w_t^{\varepsilon-1}\right)^{-\mu}}{G_t\left(1+\varphi w_t^{\varepsilon-1}\right)^{-1} + (1-G_t)\left(1+\varphi w_t^{\varepsilon-1}\right)^{-\mu}}\right)^{\frac{1}{1+q}}, \tag{90}$$

which together with (11) determines v_t and w_t for given $\left(N_t, G_t, H_t^P\right)$. From then on the reasoning follows that of Appendix 8.2.1. First, we derive that $w_\infty > 0$, such that $g_\infty^v = g_\infty^{GDP} = \psi g_\infty^N$, and that we must have $g_\infty^w < g_\infty^v$, such that $\bar{j}_\infty = 0$. Second, we study the asymptotic behavior of w_t both when $\varepsilon < \infty$ and when $\varepsilon = \infty$.

Case with $\varepsilon < \infty$. Plugging (90) in (11) gives w_t in function of N_t, G_t and H_t^P:

$$w_t = \frac{\frac{\sigma-1}{\sigma}\beta^{\frac{1+\beta q}{1+q}}\left((1-\beta)\frac{1+q}{q}\right)^{\frac{(1-\beta)q}{1+q}}\frac{1}{l^{1-\beta}}\left(\frac{H_t^P}{\overline{H}}\right)^{\frac{1-\beta}{1+q}}N^{\frac{1}{\sigma-1}}}{\times\frac{\left(G_t\left(1+\varphi w_t^{\varepsilon-1}\right)^{\mu-1}+(1-G_t)\right)^{\frac{1-\beta}{1+q}}}{\left(G_t(\varphi w^{\varepsilon-1}+1)^{\mu}+(1-G_t)\right)^{\frac{1}{1-\sigma}+\frac{1-\beta}{1+q}}}}, \tag{91}$$

which replaces (43). It is direct that when $G_\infty < 1$, we obtain (26). In this case, we further have

$$g_\infty^{\bar{j}} = q\left(g_\infty^w - g_\infty^v\right) = -\frac{q\beta\left(\sigma-1\right)}{1+q+\beta\left(\sigma-1\right)}g_\infty^{GDP}. \tag{92}$$

Case with $\varepsilon = \infty$. In this case, (91) becomes

$$w_t = \frac{\frac{\sigma-1}{\sigma}\beta\left(\frac{1+q}{q}\right)^{(1-\beta)q}\left(\frac{H_t^P}{l\overline{H}}\right)^{1-\beta}N^{\frac{1}{\sigma-1}}\left(1-G_t\right)^{\frac{1-\beta}{1+q}}}{\times\left(G_t\left(\widetilde{\varphi}w\right)^{\beta(\sigma-1)}+(1-G_t)\right)^{\frac{1}{\sigma-1}-\frac{1-\beta}{1+q}}}, \text{ if } w_t > \widetilde{\varphi}^{-1},$$

$$w_t = \frac{\sigma-1}{\sigma}\beta\left(\frac{1+q}{q}\right)^{(1-\beta)q}\left(\frac{H_t^P}{l\overline{H}}\right)^{1-\beta}N^{\frac{1}{\sigma-1}}, \text{ if } w_t < \widetilde{\varphi}^{-1}.$$

Once again, following the steps of Appendix 8.2.1, we get that if $G_\infty < 1$, (26) applies (and accordingly we also get (92)).

8.7.2 Dynamic system

It is convenient to redefine $n_t \equiv N_t^{\frac{-\beta}{(1-\beta)}\frac{(1+q)}{1+q+\beta(\sigma-1)}}$, we can then write the entire dynamic system as a system of differential equations in $\left(n_t, G_t, \widehat{h}_t^A, \chi_t\right)$ with two auxiliary variables ω_t and $\bar{j}_t$. Equations (27) is now given by

$$\dot{n}_t = -\frac{\beta}{1-\beta}\frac{1+q}{1+q+\beta\left(\sigma-1\right)}g_t^N n_t,$$

(28), (30), (31), (35), (37) still apply and equation (38) as well provided that H is replaced by H_t given by (88). ω_t is implicitly defined by:

$$\omega_t = n_t\left(\begin{array}{c}\left(\frac{\sigma-1}{\sigma}\right)^{\frac{1+q}{1-\beta}}\beta^{\frac{1+\beta q}{1-\beta}}\left((1-\beta)\frac{1+q}{q}\right)^{q}\frac{H_t^P}{l^{1+q}\overline{H}}\left(G_t\left(1+\varphi\omega_t^{-\frac{1}{\mu}}\right)^{\mu-1}+(1-G_t)\right)\\ \times\left(G_t\left(\varphi+\omega_t^{\frac{1}{\mu}}\right)^{\mu}+(1-G_t)\,\omega_t\right)^{\psi(1+q)-1}\end{array}\right)^{\frac{\beta(1-\sigma)}{1+q+\beta(\sigma-1)}},$$

which replaces (39) and is a rewriting of (91) and $\bar{j}_t$ is given by

$$\bar{j} = \left(\omega_t \frac{q}{1+q} \frac{\beta}{1-\beta} \frac{G_t \left(1 + \varphi \omega_t^{-\frac{1}{\mu}}\right)^{\mu-1} + 1 - G_t}{G_t \left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} + (1-G_t)\,\omega_t} \frac{H_t^P}{\overline{H}} \right)^{\frac{q}{1+q}}$$

which is derived using (89) and (90).

The steady state for this system involves $n^* = \omega^* = 0$ and therefore $j^* = 0$. As a result $H^* = \overline{H}$, so that the steady state values of $\left(g^{N*}, G^*, \widehat{h}^{A*}, \chi^*\right)$ are identical to the baseline case with $\overline{H}$ replacing H.

8.8 Alternative production technology for machines

We now assume that the high-skill factor share in inputs production is higher for machines production than it is for the production of the final good. The analysis follows similar steps as in the baseline model. The cost function (4) now becomes

$$c_k\left(\alpha\left(i\right)\right) = \beta_k^{-\beta_\kappa}\left(1-\beta_k\right)^{-(1-\beta_k)}\left(w^{1-\varepsilon} + \varphi\left(p^x\right)^{1-\varepsilon}\alpha\left(i\right)\right)^{\frac{\beta_k}{1-\varepsilon}} v^{1-\beta_k}, \tag{93}$$

for $k \in \{1,2\}$ indexing, respectively, the production of final good and machines.

As before aggregating (93) and the price normalization gives a "productivity" condition, which replaces (11).

$$\left(G\left(w^{1-\epsilon} + \varphi\left(p^x\right)^{1-\epsilon}\right)^{\mu_1} + (1-G)w^{\beta_1(1-\sigma)}\right)^{\frac{1}{1-\sigma}} v^{1-\beta_1} = \frac{\sigma-1}{\sigma}\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1} N^{\frac{1}{\sigma-1}}, \tag{94}$$

where we generalize the definition of μ: $\mu_k \equiv \frac{\beta_k(\sigma-1)}{\varepsilon-1}$. Following the same methodology for the production of machines, we get

$$\left(G\left(w^{1-\epsilon} + \varphi\left(p^x\right)^{1-\epsilon}\right)^{\mu_2} + (1-G)w^{\beta_2(1-\sigma)}\right)^{\frac{1}{1-\sigma}} v^{1-\beta_2} = \frac{\sigma-1}{\sigma}\beta_2^{\beta_2}(1-\beta_2)^{1-\beta_2} N^{\frac{1}{\sigma-1}} p^x. \tag{95}$$

Taking the ratio between these two expressions, we get

$$\frac{\left(G\left(\left(\frac{w}{p^x}\right)^{1-\epsilon}+\varphi\right)^{\mu_2}+(1-G)\left(\frac{w}{p^x}\right)^{\beta_2(1-\sigma)}\right)^{\frac{1}{1-\sigma}}\upsilon^{\beta_1-\beta_2}}{\left(G\left(\left(\frac{w}{p^x}\right)^{1-\epsilon}+\varphi\right)^{\mu_1}+(1-G)\left(\frac{w}{p^x}\right)^{\beta_1(1-\sigma)}\right)^{\frac{1}{1-\sigma}}}=\frac{\beta_2^{\beta_2}(1-\beta_2)^{1-\beta_2}(p^x)^{1-\beta_2+\beta_1}}{\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}}. \tag{96}$$

The share of revenues accruing to machines in the production of intermediate input i for the usage-k (i.e for use in the final sector or the machines sector) is given by

$$\nu_{k,x}(\alpha(i))=\frac{\sigma-1}{\sigma}\alpha(i)\beta_k\frac{\varphi(p^x)^{1-\varepsilon}}{w^{1-\varepsilon}+\varphi(p^x)^{1-\varepsilon}}, \tag{97}$$

aggregating over all intermediates inputs and denoting $R_k(\alpha(i))$ the revenues generated through usage k by a firm of type $\alpha(i)$, we get that the total expenses in machines are given by

$$p^x X=NG\left(R_1(1)\nu_{1,x}(1)+R_2(1)\nu_{2,x}(1)\right). \tag{98}$$

The zero profit condition in the machines sector gives

$$p^x X=N\left(GR_2(1)+(1-G)R_2(0)\right). \tag{99}$$

Revenues themselves are given by

$$R_1(\alpha(i))=\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}c_1(\alpha(i))^{1-\sigma}Y \text{ and } R_2(\alpha(i))=\left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1}c_2(\alpha(i))^{1-\sigma}p^x X, \tag{100}$$

so that (7) still holds but separately for revenues occurring from each activity and with μ_k replacing μ. Combining (7), (97), (98) and (99), we get

$$\begin{gathered}\left(G\left(1-\frac{\sigma-1}{\sigma}\beta_2\frac{\varphi(p^x)^{1-\varepsilon}}{w^{1-\varepsilon}+\varphi(p^x)^{1-\varepsilon}}\right)+(1-G)\left(1+\varphi\left(\frac{w}{p^x}\right)^{\varepsilon-1}\right)^{-\mu_2}\right)\frac{R_2(1)}{R_1(1)}\\=G\frac{\sigma-1}{\sigma}\beta_1\frac{\varphi(p^x)^{1-\varepsilon}}{w^{1-\varepsilon}+\varphi(p^x)^{1-\varepsilon}}\end{gathered}, \tag{101}$$

which determines the revenues ratio as a function of input prices solely.

To derive low-skill wages, we compute the share of revenues accruing to low-skill labor in the production of intermediate input i for the usage-k as:

$$\nu_{k,l}\left(\alpha\left(i\right)\right)=\frac{\sigma-1}{\sigma}\beta_{k}\left(1+\alpha\left(i\right)\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-1},$$

so that total low-skill income can be written as:

$$wL=N\left(GR_{1}(1)\nu_{1,l}(1)+(1-G)R_{1}(0)\nu_{1,l}(0)+GR_{2}(1)\nu_{2,l}(1)+(1-G)R_{2}(0)\nu_{2,l}(0)\right). \tag{102}$$

The share of revenues going to high-skill workers is given by $\nu_{k,h}=\frac{\sigma-1}{\sigma}\left(1-\beta_{k}\right)$ both in automated and non-automated firms. As a result

$$vH^{P}=N\left(\nu_{1,h}\left(GR_{1}(1)+(1-G)R_{1}(0)\right)+\nu_{2,h}\left(GR_{2}(1)+(1-G)R_{2}(0)\right)\right), \tag{103}$$

Take the ratio between (102) and (103), and use (7) to obtain:

$$\frac{wL}{vH^{P}}=\frac{\left\{\begin{array}{c}\beta_{1}\left(G\left(1+\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-1}+(1-G)\left(1+\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-\mu_{1}}\right)\\+\beta_{2}\frac{R_{2}(1)}{R_{1}(1)}\left(G\left(1+\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-1}+(1-G)\left(1+\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-\mu_{2}}\right)\end{array}\right\}}{\left\{\begin{array}{c}\left(1-\beta_{1}\right)\left(G+(1-G)\left(1+\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-\mu_{1}}\right)\\+\left(1-\beta_{2}\right)\frac{R_{2}(1)}{R_{1}(1)}\left(G+(1-G)\left(1+\varphi\left(\frac{w}{p^{x}}\right)^{\varepsilon-1}\right)^{-\mu_{2}}\right)\end{array}\right\}} \tag{104}$$

Together (94), (96), (101) and (104) determine w,v,p^{x} and $R\left(2\right)/R\left(1\right)$ given N,G and H^{P}. The equations for the dynamic part are similar to the baseline model. We now prove Proposition 5 and the associated footnote.

8.8.1 Asymptotic behavior for $\varepsilon<1$

As the supply of machines is going up and there is imperfect substitutability in production between machines and low-skill labor, any equilibrium must feature $w_{\infty}/p_{\infty}^{x}=\infty$ even if $w_{\infty}<\infty$. Applying this to (96), we get

$$\left(p_{t}^{x}\right)^{1-\beta_{2}+\beta_{1}}\sim\frac{\beta_{1}^{\beta_{1}}\left(1-\beta_{1}\right)^{1-\beta_{1}}}{\beta_{2}^{\beta_{2}}\left(1-\beta_{2}\right)^{1-\beta_{2}}}\varphi^{\frac{\mu_{2}-\mu_{1}}{1-\sigma}}v_{t}^{\beta_{1}-\beta_{2}}. \tag{105}$$

Further plugging this last relationship in (94), we get:

$$v_t \sim \frac{\left(\frac{\sigma-1}{\sigma}\right)^{1+\frac{\beta_1}{1-\beta_2}} \varphi^{\psi_2\mu_1} \beta_1^{\beta_1} (1-\beta_1)^{1-\beta_1} \left(\beta_2^{\frac{\beta_2}{1-\beta_2}} (1-\beta_2)\right)^{\beta_1}}{G_t^{\psi_1\left(1+\frac{\beta_1(\beta_2-\beta_1)}{(1-\beta_2)}\right)} N_t^{\psi_1\left(1+\frac{\beta_1(\beta_2-\beta_1)}{(1-\beta_2)}\right)}}, \tag{106}$$

where for x_t and y_t (possibly with no limits), $x_t \sim y_t$ signifies $x_t/y_t \to 1$. Hence

$$g_\infty^v = \psi_1 \left(1 + \frac{\beta_1 (\beta_2 - \beta_1)}{(1-\beta_2)}\right) g_\infty^N. \tag{107}$$

Through (101), the revenues of the machines sector and the final good sector are of the same order, which implies that Y, $p^x X$ and v grow at the same rate. Therefore

$$g_\infty^{GDP} = g_\infty^Y = g_\infty^v = \psi_1 \left(1 + \frac{\beta_1 (\beta_2 - \beta_1)}{(1-\beta_2)}\right) g_\infty^N.$$

In fact (101) gives

$$\frac{R_{2,t}(1)}{R_{1,t}(1)} \sim \frac{\frac{\sigma-1}{\sigma}\beta_1}{1 - \frac{\sigma-1}{\sigma}\beta_2}. \tag{108}$$

Using (105) and (106), one further gets:

$$p_t^x \sim \frac{\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}}{\left(\beta_2^{\beta_2}(1-\beta_2)^{1-\beta_2}\right)^{\frac{1-\beta_1}{1-\beta_2}}} \varphi^{\psi_2\mu_1\frac{(\beta_1-\beta_2)}{\beta_1}} \left(\frac{\sigma}{\sigma-1}\right)^{\frac{\beta_2-\beta_1}{1-\beta_2}} G_t^{-\psi_2(\beta_2-\beta_1)} N_t^{-\psi_2(\beta_2-\beta_1)},$$

therefore

$$g_\infty^{p_x} = -\psi_2 (\beta_2 - \beta_1) g_\infty^N < 0, \tag{109}$$

since $\beta_2 > \beta_1$. Using that $w_\infty / p_\infty^x = \infty$ and (108) in (104) leads to:

$$w_t \left(\frac{w_t}{p_t^x}\right)^{\varepsilon-1} \sim \frac{v_t H_t^P \left(\begin{array}{c} \beta_1 \left(G_t + (1-G_t)\left(\varphi\left(\frac{w_t}{p_t^x}\right)^{\varepsilon-1}\right)^{1-\mu_1}\right) \\ +\beta_2 \frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2} \left(G_t + (1-G_t)\left(\varphi\left(\frac{w_t}{p_t^x}\right)^{\varepsilon-1}\right)^{1-\mu_2}\right) \end{array}\right)}{\varphi G_t L \left(1 - \beta_1 + (1-\beta_2)\frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2}\right)}. \tag{110}$$

Since $\beta_2 > \beta_1$, then $(1-G_t)\left(\frac{w_t}{p_t^x}\right)^{(\varepsilon-1)(1-\mu_1)}$ dominates $(1-G_t)\left(\frac{w_t}{p_t^x}\right)^{(\varepsilon-1)(1-\mu_2)}$ asymptotically regardless of the value of G_∞ (in other words, we can always ignore $(1-G_t)\left(\frac{w_t}{p_t^x}\right)^{(\varepsilon-1)(1-\mu_2)}$ in our analysis).

The reasoning then follows that of Appendix 8.2.1. If $G_\infty < 1$, then (110) implies

$$w_t^{1+\beta_1(\sigma-1)} \sim (p_t^x)^{(\sigma-1)\beta_1} \frac{vH^P\beta_1(1-G_t)}{\varphi^{\mu_1}G_tL\left(1-\beta_1+(1-\beta_2)\frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2}\right)}, \tag{111}$$

which, together with (107) and (109) gives (41).

Alternatively assume that $G_\infty = 1$ and that $\lim(1-G_t)N_t^{\psi_2(1-\mu_1)\frac{\varepsilon-1}{\varepsilon}}$ exists and is finite. Suppose first that $\limsup(1-G_t)\left(\frac{w_t}{p_t^x}\right)^{(\varepsilon-1)(1-\mu_1)} = \infty$, then there must be a subsequence where (111) is satisfied, which with (107) and (109) leads to a contradiction with the assumption that $\lim(1-G_t)N_t^{\psi_2(1-\mu)\frac{\varepsilon-1}{\varepsilon}}$ exists and is finite.

If $\lim(1-G)\left(\frac{w_t}{p^x}\right)^{(\varepsilon-1)(1-\mu_1)} = 0$, then (110) gives

$$w_t^\varepsilon \sim \frac{(p_t^x)^{\varepsilon-1}v_tH_t^P\left(\beta_1+\beta_2\frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2}\right)}{\varphi L\left(1-\beta_1+(1-\beta_2)\frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2}\right)},$$

which implies with (107) and (109) that:

$$g_\infty^w = \frac{1}{\varepsilon}\left(1-\frac{(\beta_2-\beta_1)(\varepsilon-1)}{(1-\beta_2+\beta_1)}\right)g_\infty^{GDP}. \tag{112}$$

Finally, if $\limsup(1-G_t)w_t^{(\varepsilon-1)(1-\mu)}$ is finite but strictly positive, then as in Appendix 8.2.1, one can show that this requires that $\lim(1-G_t)N_t^{\frac{\psi_2}{\varepsilon}(\varepsilon-1)(1-\mu_1)} > 0$, from which we can derive that (112) also holds in that case. This proves Proposition 5 and the associated footnote in the imperfect substitutes case.

8.8.2 Perfect substitutes case

In the perfect substitutes case, (94) becomes:

$$\begin{array}{c}\left(G\widetilde{\varphi}^{\beta_1(\sigma-1)}(p^x)^{\beta_1(1-\sigma)}+(1-G)w^{\beta_1(1-\sigma)}\right)^{\frac{1}{1-\sigma}}v^{1-\beta_1} \\ = \frac{\sigma-1}{\sigma}\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}N^{\frac{1}{\sigma-1}} \text{ for } w > p^x/\widetilde{\varphi}\end{array}, \tag{113}$$

$$w^{\beta_1}v^{1-\beta_1}=\frac{\sigma-1}{\sigma}\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}N^{\frac{1}{\sigma-1}}\text{ for }w<p^x; \tag{114}$$

(96) becomes

$$\frac{\left(G\widetilde{\varphi}^{\beta_2(\sigma-1)}\left(p^x\right)^{\beta_2(1-\sigma)}+(1-G)w^{\beta_2(1-\sigma)}\right)^{\frac{1}{1-\sigma}}v^{\beta_1-\beta_2}}{\left(G\widetilde{\varphi}^{\beta_1(\sigma-1)}\left(p^x\right)^{\beta_1(1-\sigma)}+(1-G)w^{\beta_1(1-\sigma)}\right)^{\frac{1}{1-\sigma}}}=\frac{\beta_2^{\beta_2}(1-\beta_2)^{1-\beta_2}p^x}{\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}}\text{ for }w>p^x/\widetilde{\varphi}, \tag{115}$$

$$p_x=\frac{\beta_1^{\beta_1}(1-\beta_1)^{1-\beta_1}}{\beta_2^{\beta_2}(1-\beta_2)^{1-\beta_2}}w^{\beta_2-\beta_1}v^{\beta_1-\beta_2}\text{ for }w<p^x/\widetilde{\varphi}; \tag{116}$$

(101) becomes

$$\left(G\left(1-\frac{\sigma-1}{\sigma}\beta_2\right)+(1-G)\,\widetilde{\varphi}^{\beta_2(1-\sigma)}\left(\frac{w}{p^x}\right)^{\beta_2(1-\sigma)}\right)\frac{R_2\left(1\right)}{R_1(1)}=G\frac{\sigma-1}{\sigma}\beta_1\text{ for }w>p^x/\widetilde{\varphi}, \tag{117}$$

with $R_2\left(1\right)=0$ for $w<p^x/\widetilde{\varphi}$; and (104) becomes

$$\frac{wL}{vH^P}=(1-G)\frac{\beta_1\left(\widetilde{\varphi}\frac{w}{p^x}\right)^{\beta_1(1-\sigma)}+\beta_2\frac{R_2(1)}{R_1(1)}\left(\widetilde{\varphi}\frac{w}{p^x}\right)^{\beta_2(1-\sigma)}}{\left\{\begin{array}{c}(1-\beta_1)\left(G+(1-G)\left(\widetilde{\varphi}\frac{w}{p^x}\right)^{\beta_1(1-\sigma)}\right)\\+(1-\beta_2)\frac{R_2(1)}{R_1(1)}\left(G+(1-G)\left(\widetilde{\varphi}\frac{w}{p^x}\right)^{\beta_2(1-\sigma)}\right)\end{array}\right\}}\text{ for }w>p^x/\widetilde{\varphi}, \tag{118}$$

$$\frac{wL}{vH^P}=\frac{\beta_1}{1-\beta_1}\text{ for }w<p^x/\widetilde{\varphi}. \tag{119}$$

Together (114), (116) and (119) show that we must have $w_t\geq\frac{p_t^x}{\widetilde{\varphi}}$ for t large enough, which delivers (107) and (109).

Assume that $G_\infty<1$, then (118) gives (111) from which we get that (41) is satisfied.

Now consider the case where $G_\infty=1$ and $\lim\left(1-G_t\right)N_t^{\psi_2}$ exists and is finite. Then (118) and (117) imply

$$w_t\sim\left(1-G_t\right)v_t\left(\widetilde{\varphi}\frac{w_t}{p_t^x}\right)^{\beta_1(1-\sigma)}\frac{\left(\beta_1+\beta_2\frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2}\left(\widetilde{\varphi}\frac{w_t}{p_t^x}\right)^{-(\beta_2-\beta_1)(\sigma-1)}\right)}{1-\beta_1+(1-\beta_2)\frac{\frac{\sigma-1}{\sigma}\beta_1}{1-\frac{\sigma-1}{\sigma}\beta_2}}\frac{H_t^P}{L}\text{for }w>p^x/\widetilde{\varphi}.$$

We can then derive that $\frac{\widetilde{\varphi} w_t}{p_t^x}$ must have a finite (and positive) limit, so that

$$g_\infty^w = g_\infty^{p^x} = -\frac{\beta_2 - \beta_1}{1 - \beta_2 + \beta_1} g_\infty^{GDP}.$$

This proves Proposition 5 and its associated footnote in the perfect substitutes case.

8.9 Machines as a capital stock

The solution follows similar steps to the baseline case. We denote by $\widetilde{r}_t$ the gross rental rate of machines and by Δ their depreciation rate, such that:

$$\widetilde{r}_t = r_t + \Delta. \tag{120}$$

The unit cost of intermediate input i is now given by

$$c\left(w, v, \widetilde{r}, \alpha\left(i\right)\right) = \beta^{-\beta}\left(1-\beta\right)^{-(1-\beta)}\left(w^{1-\varepsilon} + \varphi \widetilde{r}^{1-\varepsilon}\right)^{\frac{\beta}{1-\varepsilon}} v^{1-\beta}. \tag{121}$$

instead of (4). We can then derive a productivity condition:

$$\left(G\left(\varphi \widetilde{r}^{1-\varepsilon} + w^{1-\varepsilon}\right)^{\mu} + \left(1-G\right) w^{\beta(1-\sigma)}\right)^{\frac{1}{1-\sigma}} v^{1-\beta} = \frac{\sigma - 1}{\sigma}\beta^{\beta}\left(1-\beta\right)^{1-\beta} N^{\frac{1}{\sigma-1}}, \tag{122}$$

a skill premium condition:

$$\frac{v}{w} = \frac{1-\beta}{\beta}\frac{L}{H^P}\frac{G + \left(1-G\right)\left(1 + \varphi\left(\frac{w}{\widetilde{r}}\right)^{\varepsilon-1}\right)^{-\mu}}{G\left(1 + \varphi\left(\frac{w}{\widetilde{r}}\right)^{\varepsilon-1}\right)^{-1} + \left(1-G\right)\left(1 + \varphi\left(\frac{w}{\widetilde{r}}\right)^{\varepsilon-1}\right)^{-\mu}}, \tag{123}$$

and, taking the ratio of revenues going to high-skill workers in production over revenues going to machines owners, a relationship linking the gross rental rate of capital and high-skill wages:

$$\frac{v}{\widetilde{r}} = \frac{1-\beta}{\beta}\frac{K}{H^P}\frac{G\left(\varphi + \left(\frac{w}{\widetilde{r}}\right)^{1-\varepsilon}\right)^{\mu} + \left(1-G\right)\left(\frac{w}{\widetilde{r}}\right)^{\beta(1-\sigma)}}{G\varphi\left(\varphi + \left(\frac{w}{\widetilde{r}}\right)^{1-\varepsilon}\right)^{\mu-1}}. \tag{124}$$

In addition, we still have (8) and the Euler equation (22), while the capital accumulation equation is given by

$$\dot{K}_t = Y_t - C_t - \Delta K_t. \tag{125}$$

8.9.1 Proposition 2

It is direct to see that together these equations imply that if g_t^N, G_t, H_t^P admit strictly positive limits, we must have

$$g_\infty^v = g_\infty^K = g_\infty^Y = g_\infty^C = \psi g_\infty^N,$$

from which, the Euler equation (22) gives that

$$\widetilde{r}_\infty = \rho + \theta\psi g_\infty^N + \Delta.$$

Therefore, using (122) and (124), one recovers Proposition 2 in this case.

8.9.2 Transitional dynamics

The transitional dynamics can now be expressed as a system of differential equations in $\mathbf{x}_t \equiv \left(n_t, G_t, \widehat{K}_t, \widehat{h}_t^A, \widehat{v}_t, \widehat{c}_t\right)$ where $\widehat{K}_t = N_t^{-\psi}$—so that there is one additional state variable and one additional control variable. Moreover, we redefine $\omega_t \equiv \left(\frac{w}{\widetilde{r}}\right)^{\frac{1}{\beta(1-\sigma)}}$. First (27), and (28) obviously still hold. The ratio of profits is naturally now given by

$$\frac{\pi(w_t, v_t, 0)}{\pi(w_t, v_t, 1)} = \frac{\widehat{\pi}_t^N}{\widehat{\pi}_t^A} = \left(1 + \varphi\left(\frac{w_t}{\widetilde{r}_t}\right)^{\varepsilon-1}\right)^{-\mu} = \omega_t\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{-\mu}, \tag{126}$$

in addition, (17), (18), (19) and (21) still hold, so that we still get (30). Using (18), (19), (21), (120) and (126), we get:

$$\dot{\widehat{v}}_t = \widehat{v}_t\left(\widetilde{r}_t - \Delta - (\psi - 1)\, g_t^N - \gamma\omega_t\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{-\mu}\frac{\widehat{\pi}_t^A}{\widehat{v}_t} - \gamma\frac{1-\kappa}{\kappa}\widehat{h}_t^A\right). \tag{127}$$

The Euler equation (22) and the capital accumulation equation (125) can be rewritten as

$$\dot{\widehat{c}}_t = \frac{\widehat{c}_t}{\theta}\left(\widetilde{r}_t - \Delta - \rho - \theta\psi g_t^N\right), \tag{128}$$

$$\dot{\widehat{K}}_t = \sigma\psi H_t^P \widehat{v}_t - \widehat{c}_t - \left(\Delta + \psi g_t^N\right)\widehat{K}_t \tag{129}$$

where we used (120) and (34). Together (27), (28), (30), (127), (128) and (129) give a system of differential equations which depend on g_t^N, $\widetilde{r}_t$, $\widehat{\pi}_t^A/\widehat{v}_t$ and ω_t.

g_t^N is still given by (38) as a function $\left(\mathbf{x}_t, \omega_t, H_t^P\right)$, (122) implies

$$\widetilde{r}_t = \widehat{v}_t^{-\frac{1-\beta}{\beta}} \left(\frac{\sigma-1}{\sigma}\right)^{\frac{1}{\beta}} \beta\left(1-\beta\right)^{\frac{1-\beta}{\beta}} \left(G_t\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} + \left(1-G_t\right)\omega_t\right)^{\frac{1}{\beta(\sigma-1)}}, \tag{130}$$

which gives $\widetilde{r}_t$ as a function of $(\mathbf{x}_t, \omega_t)$. We still have (32), which combined with (121) gives

$$\pi_t^A = \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}\left(\beta^\beta\left(1-\beta\right)^{1-\beta}\right)^{\sigma-1}\left(\omega_t^{\frac{1}{\mu}} + \varphi\right)^{\mu} \widetilde{r}^{\beta(1-\sigma)} v^{-\psi^{-1}} Y_t. \tag{131}$$

Combining (34) with (131) and (130) gives (35) again as a function of $\left(\mathbf{x}_t, \omega_t, H_t^P\right)$. To get H_t^P as a function of $(\mathbf{x}_t, \omega_t)$, rewrite (124):

$$H_t^P = \frac{1-\beta}{\beta}\frac{\widetilde{r}_t\widehat{K}_t}{\widehat{v}_t}\frac{G_t\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu} + \left(1-G_t\right)\omega_t}{G_t\varphi\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu-1}}. \tag{132}$$

Finally, combining (123) with (124) gives an equation which implicitly defines ω_t:

$$\omega_t = n_t\left(\frac{LG_t\varphi\left(\varphi + \omega_t^{\frac{1}{\mu}}\right)^{\mu-1}}{\widehat{K}_t\left(G\left(1 + \varphi\omega_t^{-\frac{1}{\mu}}\right)^{\mu-1} + (1-G)\right)}\right)^{\frac{\beta(\sigma-1)}{1+\beta(\sigma-1)}}$$

This finishes the description of the system of differential equations that the equilibrium must satisfy.

8.9.3 steady state.

As before, we can look for a steady state of this system with $G^* > 0$ and $g^{N*} > 0$. Then, we must have $n^* = \omega^* = 0$, and (53), (54) and (55) still hold. (128) implies

$$\widetilde{r}^* = \Delta + \rho + \theta\psi g^{N*}, \tag{133}$$

while (127) gives

$$\widetilde{r}^* = \Delta + (\psi - 1)\, g^{N*} + \gamma \frac{1-\kappa}{\kappa} \widehat{h}^{A*}.$$

Combining these two equations delivers (52). Therefore, the steady state values of $G^*, \widehat{h}^{A*}$ and g^{N*} are the same as in the baseline case. It is then direct to use (133) to obtain $\widetilde{r}^*$, from which we get $\widehat{v}^*$ using (130), $\widehat{K}^*$ thanks to (132) and finally $\widehat{c}^*$ with (129).

8.10 Bayesian estimation

We employ Bayesian estimation methods. We are interested in the posterior distribution of the parameter vector b:

$$f(b|\hat{Y}^T) = \frac{f(\hat{Y}^T|b)\pi(b)}{\int_{b' \in \mathcal{B}} f(\hat{Y}^T|b')d\pi(b')},$$

where the domain of the uniform prior is given by table 4 below.

Table 4: Prior distribution on parameters: uniformly distributed on $[min, max]$.

	σ	μ	β	l	γ	$\tilde{\kappa}$	θ	η	κ	ρ	φ	q	n_0	G_0
min	3	1/2	0.1	0.5	0.15	0.01	1.5	0.2	0.01	0.015	5	0.2	0.1	0.1
max	6	0.99	0.95	1	0.5	0.99	2.5	0.5	0.99	0.07	15	0.99	10	0.8

A uniform prior guarantees that

$$f(b|\hat{Y}^T) \propto f(\hat{Y}^T|b).$$

The pdf does not have a closed-form solution and we employ Monte Carlo simulation methods. We are particularly interested in the mode of $f(b|\hat{Y}^T)$ and the unconditional posterior distributions of $f(b_i, \hat{Y}^T)$ for each parameter b_i.

8.10.1 The mode

We employ the fmincon interior-point algorithm in Matlab to find $max_b f(\hat{Y}^T(b, \hat{\mathbf{x}}(b_P))|b)$, where $\hat{\mathbf{x}}(b_P)$ is the solution to the system of differential equations and the dependence of $\hat{Y}^T$ on both b and $\mathbf{x}^*$ is made explicit (recall that b_P is the subset of the parameter vector, b, concerning the deterministic model). We employ an algebraic gradient, by noting that except for numerical and discretization errors, the solution of differential equations to the system described above, $\mathbf{x}^*$, is defined as:

$$S(\mathbf{x}^*, b) = \overrightarrow{0},$$

such that:

$$d\mathbf{x}^* = -[D_{\mathbf{x}}S(\mathbf{x}^*, b)]^{-1}D_bS(\mathbf{x}^*, b)db,$$

which—along with analytical expressions for the derivatives of $\hat{Y}^T$ and f—give an analytical expression for the gradient D_bf. We use a finite-difference Hessian. We start the algorithm at 20 points drawn from the uniform distribution on $\mathcal{B}$. We include a covariance estimate for the mode/MLE estimator, by writing the MLE maximization problem as $max_b log f(\hat{Y}^T|b) = max_b \Sigma_{m=1}^M \Sigma_{t=1}^T log f(\hat{Y}_t|\hat{Y}^{t-1}, b)$, and noting that the maximum likelihood estimator is defined as $\Sigma_{m=1}^M \Sigma_{t=1}^T \frac{\partial}{\partial b} log f(\hat{Y}^t|\hat{Y}^{t-1}, b^{MLE}) = 0$. Standard asymptotics then imply that

$$\sqrt{MT}\left(b_0 - b^{MLE}\right) \xrightarrow{d} N(0, H^{-1}IH^{-1}),$$

where $I = \frac{1}{TM}\sum_m \sum_t \frac{\partial log f^t(\hat{Y}_t|\hat{Y}^{t-1}, b^{MLE})}{\partial b} \frac{\partial log f^t(\hat{Y}_t|\hat{Y}^{t-1}, b^{MLE})'}{\partial b}$ and $H = \frac{1}{TM}\sum_k \sum_t \frac{\partial^2 log f^t(\hat{Y}_t|\hat{Y}^{t-1}, b^{MLE})}{\partial b \partial b'}$.

We use the covariance estimate of the maximum likelihood estimator for the weighting function in Bayesian estimation.

8.10.2 The unconditional posterior

Let $\mathcal{B}_i$ be the domain for the parameter b_i. For such a parameter b_i we split the domain up into N non-overlapping intervals, $\mathcal{B}_{i,n}$ such that $\mathcal{B}_i = \bigcup_{n=1}^N \mathcal{B}_{i,n}$. For parameter b_i and interval n let $\mathbf{1}_{i,n}(b)$ be an indicator function for whether b_i is in interval $\mathcal{B}_{i,n}$. We are interested in:

$$\int_{b_i \in \mathcal{B}_{i,n}, b_{-i} \in \mathcal{B}_{-i}} f(b|\hat{Y}^T)db = \int_{b \in \mathcal{B}} f(b|\hat{Y}^T)\mathbf{1}_{i,n}(b)db.$$

By the law of large numbers we can then draw M values of b_m from a uniform distribution which spans $\mathcal{B}$ and use:

$$\frac{1}{M}\sum f(b_m|\hat{Y}^T)\mathbf{1}_{i,n}(b_m) \rightarrow \int_{b \in \mathcal{B}} f(b|\hat{Y}^T)\mathbf{1}_{i,n}(b)db.$$

As shown by Geweke (1989) a more efficient approach is to sample from an importance sampling density, $g(b)$ and use:

$$\frac{1}{M}\sum \frac{f(b_m|\hat{Y}^T)}{g(b_m)}\mathbf{1}_{i,n}(b_m) \to \int_{b\in\mathcal{B}} \frac{f(b|\hat{Y}^T)}{g(b_m)}\mathbf{1}_{i,n}(b)g(b_m)db = \int_{b\in\mathcal{B}} f(b|\hat{Y}^T)\mathbf{1}_{i,n}(b)db.$$

We specify $g(b)$ as a mixture density between a uniform distribution on $\mathcal{B}$ and a multivariate normal with the mode (MLE) and a diagonal covariance matrix with a variance of twice that of the covariance estimate of the maximum likelihood estimator. If a draw b does not conform to the conditions of Proposition 3 and does not have a unique asymptotic steady state the value is set at $f(b|\hat{Y}^T) = 0$ (though this does not matter in practice). The weight on the uniform distribution is 0.25. The inclusion of a uniform ensures that $f(b)/g(b)$ is close to zero when b is far away from the mode. Figure

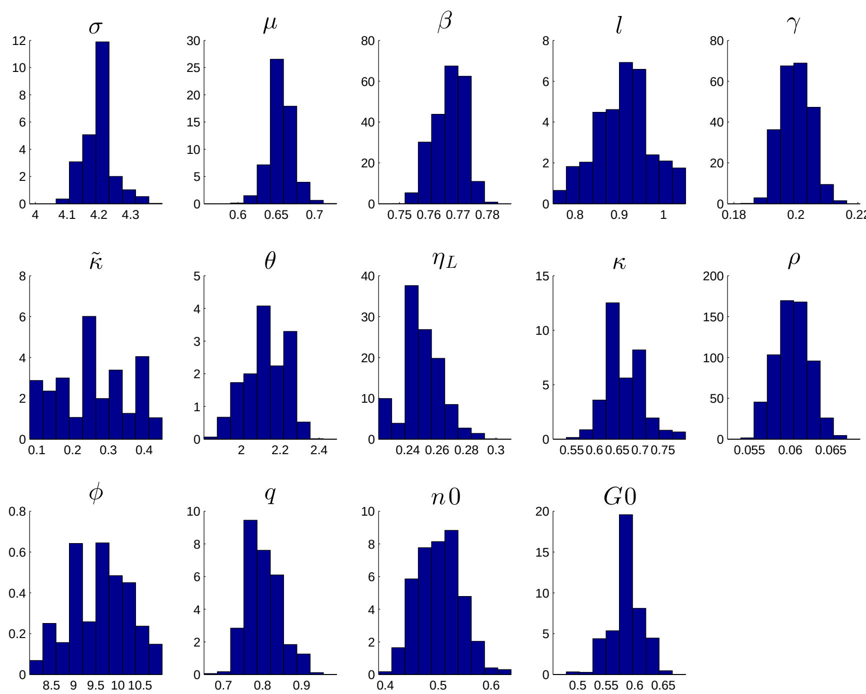

Figure 22: The unconditional posterior distributions of the model parameters, b_P.