Has the real rate of return "depreciated"?*

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Abstract

Abstract: We document that over the past half a century the average depreciation rate on capital has increased in the United States and relate this development to the long-run real rate of interest. Whereas the standard Ramsey-Cass-Koopman model finds interest rates to be independent of the depreciation rate, we show that in a broader class of models, there is a tight connection between the two and that their relationship ultimately depends on the savings motive of households. Ceteris paribus, the more important life-cycle savings are relative to bequests, the stronger is the negative effect on the long-run interest rate of an increase in the depreciation rate. We find that a plausible calibration of savings motives implies that the observed increase in the rate of depreciation can explain between 1/4 and 1/3 of the observed decline in the long-run interest rate amongst advanced economies since 1970, and perhaps 1/10th of the decline since the early 19th century.

Keywords: The Real Rate of Interest; Capital depreciation; Growth theory; Savings motive
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*The views expressed in the paper are those of the authors and do not nessesarily reflect that of the Danish Economic Councils.

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1 Introduction

Over the past fifty years, the natural real rate of interest in Advanced Economies has declined by about 300 bps (Rachel and Summers, 2019).[1] This has been accompanied by a general decline in the real rates on safe assets. At its most basic level, the observed decline can attributed to a rise in aggregate savings, a decline in aggregate investments, or a combination of the two. The list of factors that could have such an impact includes demographic movements, changes in productivity growth, changes in the income distribution, and more (a review is found in e.g. Rachel and Smith, 2017). In addition, factors that specifically influence the demand for safe assets may have had an influence (see Kiley, 2020). Rachel and Summers (2019) argue that were it not for the large increase in government debt, the decline in the real interest would have been substantially larger. We argue that a hitherto neglected factor may have contributed significantly to the decline in the real rate: A rising rate of capital depreciation. As documented below the average rate of capital depreciation appears to have risen substantially with around 1 percentage point in the United States (and likely in most advanced economies) over the last half-century.

From the standard theoretical perspective, a rising rate of capital depreciation may seem like an unlikely contributor to the secular decline in rates. Within the most popular macroeconomic framework for the analysis of long-run interest rates, the Ramsey-Cass-Koopmans (RCK) model, changes in the depreciation rate has *no* impact on the steady-state real rate of interest, which is given purely as a function of preferences and the growth rate of the economy (the Euler equation.) The same prediction emerges in the Golden Rule steady state of a Solow-Swan model. In fact, as long as the economy is not dynamically inefficient the Solow-Swan model predicts a higher rate of depreciation should work to *increase* the steady-state real rate of interest. However, if the same experiment is conducted within a Diamond OLG model, it has the opposite effect: a higher rate of capital depreciation lowers the real rate of interest on a 1:1 basis.

Why do the models provide such different answers? Given competitive markets the real rate of return equals the net marginal product of capital in all models: $r = f'(k) - \delta$ using standard notation. Therefore, all models hold the prediction that for given capital, a rise in δ lowers the net marginal product of capital. The difference lies in the subsequent adjustment in the capital stock, which depends on how aggregate savings depends on capital and labor income. In a standard Diamond model with log preferences — such that the savings rate is insensitive to the real rate of interest — capital accumulation is not affected by capital income. As a result, a higher depreciation rate leaves the long run, stock of capital unaffected, which implies that an increase in δ reduces the net marginal product of capital in the long run. Hence the prediction that the rate of depreciation lowers the long-run real rate. By way of contrast, lower capital income reduces savings in the Ramsey-Cass-Koopmans and Solow-Swan models, which implies that an increase in the depreciation rate works to lower savings. In the long-run this effect (fully) counteracts the direct influence of higher depreciation on the net gross marginal product of capital.

[1] Rachel and Summers (2019) estimate the natural real rate of interest as the interest rate that is consistent with output at its potential structural level and constant inflation.

At a deeper level the impact of factor income on aggregate savings depends on the households' savings motive. Intuitively, the key difference between the Diamond and the RCK models can be seen as whether the bequest motive is operative or not (Barro, 1974). If it is not (Diamond), capital accumulation is driven solely by wage income; in the bequest case (RCK) an economy emerges where savings are made solely from capital income along a steady-state path (e.g., Bertola, 1994).

In the analysis below, we provide an extension of a standard Diamond model where a bequest motive is present and formally demonstrate the link between the savings motive and the depreciation/interest rate nexus. In the bequest-augmented Diamond model, the marginal impact of an increase in depreciation on the steady-state real rate of interest is bounded by an interval ranging from minus 1 to (minus) the size of the labor share. The latter boundary is reached when the relative importance of the bequest motive is very large. Hence, only in the case of large bequests motives and a small labor share do we approach the Ramsey-Cass-Koopmans result of no effect of depreciation on the real rate. The reason is that, in this limit, only capital income matters for capital accumulation, exactly as is the case along a steady-state path in a Ramsey-Cass-Koopman model.

More generally, the impact of depreciation on the real rate depends on the relative importance of life cycle savings and bequest. We parameterize our model by using existing estimates of the importance of bequest in total capital, and proceed to gauge the impact of depreciation on the long-run real rate. Our analysis suggests that a rising rate of depreciation can plausibly account for a share of between 1/4 and 1/3 of the estimated decline in the natural real rate of interest since 1970.

A challenge for a number of existing explanations for the recent (post 1970) decline in the risk free rate of interest lies in accounting for the apparent fact that a secular decline has been underway for a much longer period of time. For example, recent work by Schmelzing (2019) suggest that the period since 1820 has witnessed a negative trend of about 2.29 bps per year on average. Over time horizons as long as this most theoretical determinants of the real rate receive mixed support (e.g., Borio et al. , 2017; Lunsford and West, 2019). This raises the question of whether the mechanism in focus here suffers from a similar "external validity" issue. Below we argue that it should be able to contribute albeit the quantitative significance of the mechanism is likely relatively modest.[2]

The paper is related to the recent literature that discusses plausible explanations for the observed decline in real rates of interest over the last three decades. As noted above, a number of contributing forces have been put forward. Useful overviews of the literature are found in Rachel and Smith (2017), Rachel and Summers (2019), and Kiley (2020). The contribution of the present paper is to explore the relevance of capital depreciation, a factor that seems to have been neglected so far.

Since the relative importance of bequest and life-cycle savings is key to the question at hand,

[2]The work by Schmelzing suggests this negative trend can be observed since the 15th century. We limit attention to the period after the turn of the 19th century with an eye to the theoretical frameworks used below, which probably have limited relevance for the period prior to the onset of modern growth. Back then the mechanics of the growth process were substantively different (e.g., Ashraf and Galor, 2011), which an account for a (very) long run decline in the interest rate would need to take into account.

the paper is indirectly related to the literature which has examined this question, though mainly from an empirical perspective (Kottlikoff and Summers, 1981; Modigliani, 1988; Dynan et al., 2002; Piketty, 2011). A central point made in the present paper is that establishing the relative importance of these two fundamental savings motive has important implications for the determinants of the long-run real rate of interest. Similarly related is therefore work which shows how the savings motive matters for a range of other key predictions that flow from standard growth theory. This includes Bertola (1996) on the link between the factor income distribution and growth; Galor (1996) on convergence properties (conditional vs. club convergence); Uhlig and Yanagawa (1996) on the impact of capital taxation; Jones and Manuelli (1990) on the viability of endogenous growth, and Dalgaard and Jensen (2009) on scale effects in endogenous growth models.

The paper proceeds as follows. In the following section, we document that the average rate of depreciation has gone up. Section 3 provides a brief review of standard neoclassical workhorse models concerning the nexus between depreciation and the steady-state real rate of return. Section 4 provides a hybrid model where both life cycle savings, and bequest, are operative savings motives and influence capital accumulation. In this section, we gauge the strength of the link between the real rate and the rate of depreciation. Section 5 discusses the proposed mechanism's empirical relevance, and Section 6 discusses questions raised by the present analysis and concludes.

2 Aggregate movements in the depreciation rate

The Bureau of Economic Analysis (BEA) defines depreciation as "the decline in value due to wear and tear, obsolescence, accidental damage and aging"; that is the combined depreciation of an asset from physical use as well as economic obsolescence from the introduction of new capital of higher quality or lower cost. How has the average rate of capital depreciation evolved over the last half century?

As a starting point we use data from the Penn World Table (PWT, Feenstra, Robert C., Robert Inklaar and Marcel P. Timmer, 2015) where the average depreciation rate is available. Since our focus ultimately is on the evolution the real rate of interest, which is determined by global forces, we focus on the GDP weightned average depreciation rate for the group of Advanced Economies as defined by the IMF.[3]A more specific motivation for looking at this group of countries is that they represent the sample for Rachel and Summers (2019) who recently estimate the natural rate of interest, as cited in the Introduction. Aside from the weightned average we also calculate the simple average and the median. The result is depicted in Figure 1. While the depreciation rate is relatively flat from 1970 to 1990, it has increased by about one percentage point since 1990. Since all three measures move in a similar way it is clear that the rising depreciation rate is a pervasive phenomenon.

[3]The PWT does not have information on GDP for all advanced economies throughout the time series. We, therefore, exclude the Czech Republic, Estonia, Latvia, Lithuania, San Marino, the Slovak Republic, and Slovenia from our sample. These countries constitute less than 2 per cent of GDP in 2017.

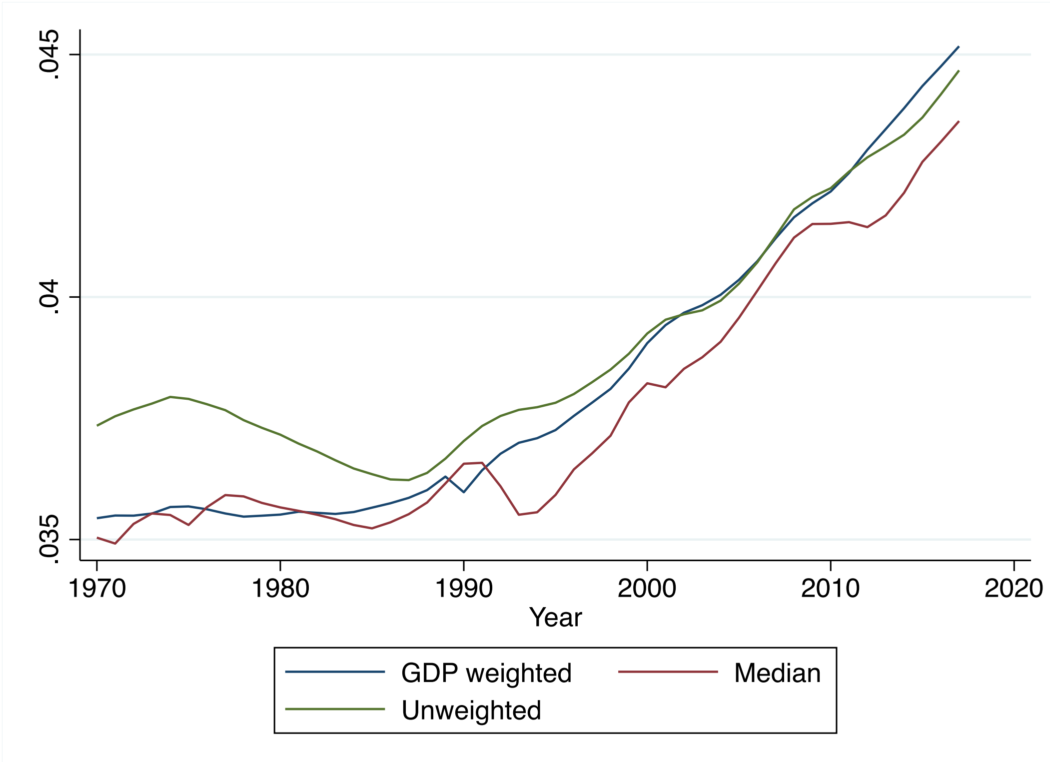


Figure 1: The depreciation rate for Advanced Economies (source: PWT)

The official data from PWT poses two potential concerns however. The first is that the depreciation rate is assumed constant over time for relatively broad capital groups (between four and eight depending on the country). If there is a shift towards more depreciation-heavy asset types within these groups, this will bias the growth in the depreciation rate downwards. Second, the depreciation rate is calculated as a weighted average where the ratio of the real capital stock of a capital type to the total stock is used as weights. Asset types with higher depreciation rates tend to have lower growth in price indices. This implies higher deflation of past real capital stocks of high-depreciation assets and consequently an increase in the depreciation rate. For instance, although ICT was several percentages of overall investment in the 1970s and its depreciation rate consequently relevant, measured by prices of today the real stock of ICT capital back then was infinitesimal and when measured from the perspective of today the weight of ICT assets in the 1970s was close to zero.[4]

[4]More formally, consider an economy with two types of assets: an ICT stock that has declining prices and a high depreciation rate, and a real estate stock that has constant prices and low depreciation rate. Let the economy be in a steady-state where investment as a share of total output is constant for both types of assets. A steady state is characterized by constant interest rates and marginal return on capital and consequently the economy faces the same depreciation rate each period. However, if the depreciation rate is measured using the real stock of capital it will mechanically feature a rising measured depreciation rate even when the economy is in a steady state: From the perspective of time s, we measure the aggregate depreciation rate at time t using the real stock of capital deflated by time s prices. Since prices decline for ICT but not for real estate this will mechanically mean that the weight of ICT declines when t is earlier. In the limit of $t \rightarrow -\infty$ the aggregate deprecation rate will be entirely that of the real estate stock and consequently lower than at time s. With lower past depreciation rates we measure an increase in the depreciation eventhough, by construction, the

Ideally one would like to assess how changing these assumption would affect the results from Figure 1. This is not possible in general, but one can provide a sense of what one would find if one is willing to limit attention to the US. Aside from being available in PWT, average depreciation can also be assessed via Euklems (Stehrer, R., A. Bykova, K. Jäger, O. Reiter and M. Schwarzhappel, 2019) and the Bureau of Economic Analysis (BEA). From the EUklems data base we obtain data on nominal stock of capital for nine asset types with corresponding (constant) depreciation rates and calculate the weighted depreciation rate for the United States. In this manner we can assess the consequences of changing the weights. The BEA relies on geometric depreciation rates and measures the depreciation rate partly through used-asset prices (Hulten and Wukoff, 1981 and Fraumani, 1997). They employ a large number of asset-specific depreciation rates as well as estimated lifetimes for assets. They allow for some updating of the depreciation rate even within these asset types. As a result, using BEA data we can relax both assumptions in the PWT.

Figure 2 shows the results. The PWT series for the US is very similar to the series for the group of Advanced Economies as a whole. But when we rely on the EUklems data, on the other hand, the depreciation rate no longer shows the same strong upward trend. This suggests that the type of bias mentioned above appears to be operative in practise. Finally, when we move to the BEA data, which in addition allows changes in depreciation within asset types the trend reemerges. Clearly, the timing of the rise differs: In PWT the increase occurs after 1990 whereas it largely takes place after 1980 in the BEA data. But these patterns suggest that the general tendency towards greater average depreciation is a robust phenomenon. Assuming the trend-similarity between the group of Advanced Economies and the US oberved in the PWT also carries over if the BEA was more widely used, data suggests that average depreciation has increased by about one percent since 1970.

Why has the depreciation rate increased? A plausible hypothesis is that the economy relies more on ICT and other high-depreciation assets and the effect comes from a reallocation towards such assets. This in turn might come about because of sectoral shifts in the economy over the past decades. We explore this using detailed data on asset types and sectors. Table 1 provides the data for the four aggregate asset types and does not show a strong reallocation across asset types. Although there has been an increase in the importance of Intellectual property with the highest depreciation rate, equipment as a whole has declined. At the same time, all four categories have seen an increase in the depreciation rate during this period. Naturally, reallocations might have happened within these categories.

To explore these issues more carefully we use the detailed subdivision of 72 asset types, each with a corresponding level of depreciation. We next calculate the overall depreciation rate as $\delta_t = \sum_i \delta_{i,t} K_{i,t} / K_t$, where $K_{i,t}$ is nominal (current-cost) stock of private fixed assets (with K_t being total stock) and $\delta_{i,t}$ is the depreciation rate (deprecation divided by stock of capital for each asset type). Finally, we follow Autor, Dorn, Katz, Patterson and Van Reenen (2020) and perform a

economy is in a steady state and consequently the environment of the agents is unchanging. We develop this argument fully in Appendix A.

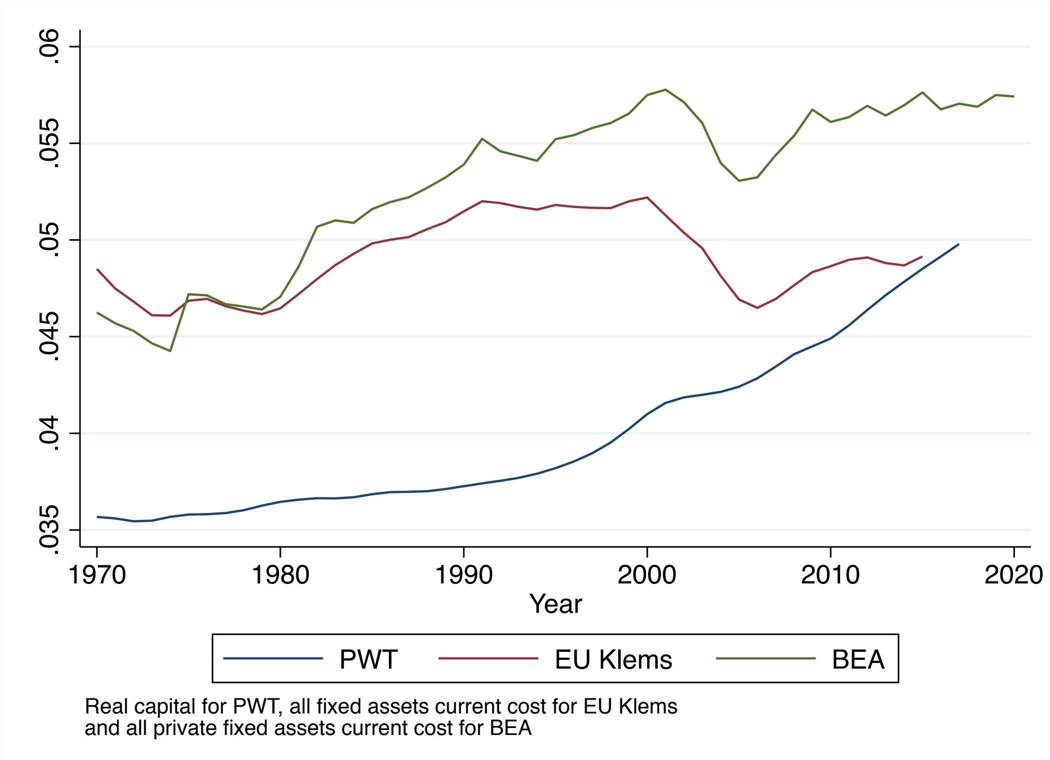
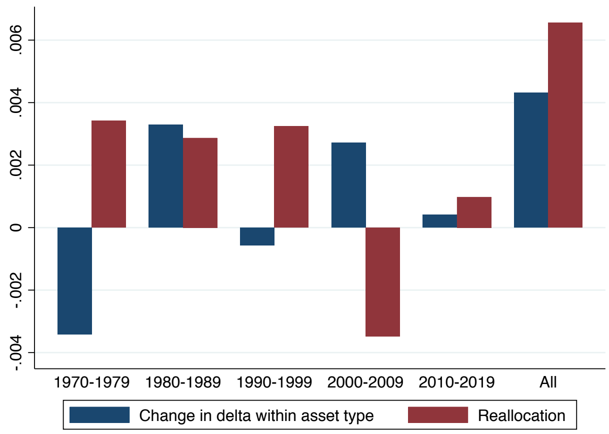


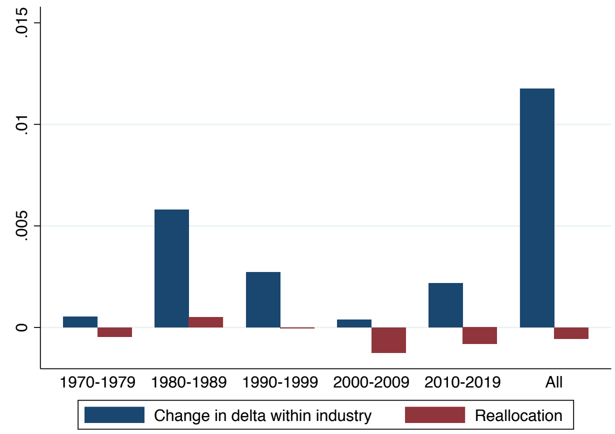
Figure 2: The deprecation rate for the United States - measured by PWT (real stock of capital), Euklems (nominal) and BEA (nominal)

	Depreciation rate		Nominal Capital (share of total)	
Year	1970	2020	1970	2020
Equipment	12.06%	13.69%	18.6%	14.2%
Intellectual Property	18.97%	24.14%	3.65%	7.4%
Residential Structures	1.92%	2.33%	45.23%	47.6
Non-Residential Structures	2.55%	2.96%	32.5%	30.7
Total	4.63%	5.74%		

Table 1: United States distribution of depreciation across asset types (BEA data)



(a) Across asset types



(b) Across industries

Figure 3: Decomposition of the depreciation rate for the United States across asset types and industries

decomposition of the change in the deprecation rate between period $t-1$ and t as:

$$\delta_t - \delta_{t-1} = \sum_i \underbrace{\delta_{i,t-1}\left(\frac{K_{i,t}}{K_t} - \frac{K_{i,t-1}}{K_{t-1}}\right)}_{Reallocation} + \sum_i \underbrace{K_{i,t}\left(\delta_{i,t} - \delta_{i,t-1}\right)}_{Change\ within\ asset\ type},$$

where the first term gives changes in the aggregate depreciation rate from differential growth in asset classes with different depreciation rates, and the second term considers changes in the depreciation rate within asset types.

Figure 3 performs this decomposition for each one of the decades from 1970 to 2019 as well as the overall time period. Though there have been substantial changes over the decades, the two terms are of roughly equal importance. Since the 72 asset types used for the calculations here are themselves aggregation of finer asset types, this estimate presents a lower bound on what the reallocation effect would be from a finer disaggregation.

Are these movements due to shifts towards sectors that rely more high-depreciation assets? Panel B of Figure 3 shows an analogous decomposition across 18 industries. The figure clearly shows that the entire change comes from within-industry changes with a minor negative contribution from reallocation. Hence, within sectors depreciation rates have been going up, partly because of a change in the composition of asset types but also because of rising depreciation within asset categories. The remainder of the present study ponders the consequences of rising depreciation for the long run real rate of interest.

3 Depreciation and the real rate: A review

3.1 The Solow-Swan model

Consider a Solow-Swan model where production is Cobb-Douglas, $Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$. Time is discrete. Since savings are generated as a constant fraction (s) of total income, and the economy is closed, $K_{t+1} = sY_t + (1-\delta) K_t$. Population grows at the constant rate n and technological progress occurs at the rate g. In the steady state the model predicts that the capital-output ratio converges to

$$\left(\frac{k}{y}\right)^* = \frac{s}{n+g+\delta}$$

where k and y are both expressed in efficiency units ($x = X/AL$).

Assuming competitive markets, the real rate of return is given by the first order condition from firm profit maximization

$$r_t = \alpha \frac{y_t}{k_t} - \delta. \tag{1}$$

In the steady state the real rate of interest therefore approaches

$$\lim_{t \to \infty} r_t = r^* = \alpha \frac{n+g+\delta}{s} - \delta. \tag{2}$$

As can be seen, the impact on r^* from a change in the depreciation rate depends on the ratio of α to s. If $\alpha = s$, which means the economy is at the Golden Rule steady state, the steady-state real rate of return is exactly independent of the depreciation rate. Only if $\alpha < s$, i.e. if the economy is dynamically inefficient, does an increase in capital depreciation lower r^*. Absent dynamic inefficiency, a Solow-Swan model would therefore predict that a higher depreciation rate will work to *increase* the real rate of interest; in the Golden-Rule steady-state $\partial r^* / \partial \delta = 0$ as $s = \alpha$.

The reason is simple. Depreciation has two separate effects on r. First, an increase in δ directly lowers the net marginal product of capital. Second, it indirectly reduces the long-run capital-output ratio and thus increases the long-run gross marginal product. In the golden rule steady state, the two countervailing forces exactly offset, whereas for a dynamically efficient economy, the reduction in the long-run capital-output ratio dominates.

3.2 The Ramsey-Cass-Koopmans model

Suppose now that consumers behave differently. Instead of being characterized by "rule-of-thumb" behavior, as in the Solow-Swan model, assume they exhibit dynastic behavior, which in effect means savings are based on a bequest motive (Barro, 1974). That is, they act as an infinitely lived agent (Ramsey, 1929; Cass, 1965; Koopmans, 1965). Without loss, assume per period utility is logarithmic for simplicity. In the Ramsey-Cass-Koopmans (RCK) model the real rate is pinned down by the first order condition assigned to the problem of maximizing utility from consumption over an infinite horizon, the consumption Euler:

$$\frac{c_{t+1}}{c_t} = r_t - \rho \Rightarrow r^* = g + \rho. \quad (3)$$

Clearly, in a RCK model, δ does not affect r^*.

The reason is that along a steady state path, the Ramsey consumer reacts to a higher depreciation rate by adjusting savings downward. As a result, a change in δ leaves the net marginal product of capital constant and with it r^*. The RCK framework thus holds the same steady state prediction as a Solow-Swan model, evaluated at the golden-rule steady state.

It is worth observing that the RCK model does not suggest δ is completely irrelevant, as changes in δ will affect r in transition. Suppose, for example, that δ rises from δ_0 to $\delta_1 > \delta_0$ in a RCK model. This will lower the long-run capital stock in efficiency units. However, in the short run, where the capital stock is fixed, the real rate of return is predicted to fall, after which it increases monotonically in transition to the new steady state where its level is the same as before the shock. The same type of effect can be recovered within a Solow-Swan model. Hence, taking transitional dynamics into account, the RCK model does leave a modest role for changes in depreciation to affect the evolution of the real rate of interest over the long run, as long as transitions are sufficiently lengthy.[5]

3.3 The Diamond model

Finally, consider a standard Diamond (1965) model. As in the analysis above, we assume production is Cobb-Douglas. Optimal savings of the young is given by $s_t w_t$ where $s_t = 1/(2+\rho)$. The savings rate is independent of the interest rate due to logarithmic preferences, an assumption we will keep to focus the presentation on the crucial differences between the Diamond and RCK models. We comment on how the results are modified with more general preferences below. The capital stock in period $t+1$ is given by the savings of the young in period t, $K_{t+1} = s_t w_t L_t$. As a result, the law of motion for capital per efficiency unit of labor, $k \equiv K/(AL)$, takes the form

$$k_{t+1} = \frac{(1-\alpha)}{(1+g)(1+n)(2+\rho)} k_t^{\alpha}, \quad (4)$$

where we have used that $w_t = (1-\alpha)k_t^{\alpha}$ from the firm first order conditions. In the steady state (which is unique and stable) the capital-output ratio approaches:

$$\left(\frac{k}{y}\right)^* = \frac{1-\alpha}{(1+g)(1+n)(2+\rho)}.$$

It is important to recognize why the capital stock, and with it output, remains unchanged by higher

[5]A standard parameterization of the RCK model would suggest that the economy closes about 4 - 5% of the gap to the steady state per year (e.g., Barro and Sala-i-Martin, 2004), implying that half the gap to the steady state is closed within 14 to 18 years. But since the real rate appears to decline systematically over the long run transitional dynamics are unlikely to be able to fully account for the data.

δ. The capital stock in period $t+1$ is purely a function of savings by the young in period t which depends on wage income i period t. Since the capital stock is productive before any depreciation happens, a higher depreciation rate only translates into a lower capital income which flows to the (non-saving) old in period t.

Substituting for $(k/y)^*$ in the first order condition for capital (equation 1) we obtain the prediction for the long run real rate of interest:

$$r^* = \alpha \frac{(1+g)(1+n)(2+\rho)}{1-\alpha} - \delta. \tag{5}$$

In this version of the Diamond model, a higher δ does not affect the gross marginal product in the steady state, and therefore translates 1:1 into a *lower* r^*. Observe that this result holds whether the economy is dynamically inefficient or not.

If we allow for more general preferences such that the savings rate of the young depends on the real rate of return, the "1:1" correspondence no longer holds, but the result still holds qualitatively. Suppose per period preferences are more general CES $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$. If so, then the savings rate will be increasing in the (expected) real rate of interest as long as the substitution effect dominates - if $\theta < 1$ - which is sufficient for the existence and uniqueness of a self-fulfilling expectation path in the model (cf Galor and Ryder, 1989, Lemma 1). As shown in Appendix B, the resulting model predicts that a higher depreciation rate lowers the steady-state capital-output ratio, but it nevertheless unambiguously also lowers the steady-state real rate of return; capital does not decline sufficiently to compensate for a higher δ. Intuitively, the greater the decline in the savings rate when the real rate falls (i.e., the smaller the coefficient of relative risk aversion, θ), the smaller the impact of depreciation on the steady-state real rate. Chetty (2006) argued that $\theta \approx 1$ (log utility) seems to be empirically reasonable. In the remaining, we thus maintain the log utility assumption.

Aside from the link between depreciation and the real rate of interest, another steady-state implication is worth noting. Over the last generation, a gap has opened up between the marginal product of capital and the risk-free return. Currently, key explanations for this fact involves rising market power, unmeasured intangibles, and risk premia (e.g., Farhi and Gourio, 2018). The Diamond model provides an additional force: rising depreciation. The standard way to calibrate the marginal product is to use the share of capital in national accounts and the observed capital-output ratio to back out the implicit marginal product. Seen through the lens of the model, the calculation is

$$\alpha \frac{Y}{K} = \frac{(r+\delta)K}{Y} \frac{Y}{K} = MPK$$

where MPK is $\frac{\alpha}{1-\alpha}(1+g)(1+n)(2+\rho)$ in steady state. Accordingly, if δ rises the gap between r^* and MPK^* widens.

4 Growth when life cycle savings and bequest both matter

4.1 The model

In this section, we employ a model where savings is the result of both life-cycle savings and bequests. Using data on the empirical importance of bequests in accounting for savings, this framework allows use to gauge the likely quantitative significance of a rising depreciation rate on the steady state real interest rate.

Consider therefore a bequest augmented Diamond model where household derive utility from passing on bequest to the next generation:[6]

$$U(c_{1t}, c_{2t+1}, b_{t+1}) = \ln c_{1t} + \frac{1}{1+\rho}(\ln c_{2t+1} + \eta \ln b_{t+1}),$$

and where η parameterizes the strength of the bequest motive. The budget constraints are

$$c_{1t} + s_t = \frac{b_t}{1+n} + w_t \quad \wedge \quad c_{2t+1} = (1 + r_{t+1})s_t - b_{t+1},$$

where received bequests are adjusted for the fact that each parent is assumed to have $1+n$ descendants. The first order conditions are:

$$b_{t+1} = \eta c_{2t+1} \tag{6}$$

$$c_{2t+1} = \frac{1 + r_{t+1}}{1+\rho} c_{1t} \tag{7}$$

Equation (6) equates marginal utilities from consuming during old age and from bequest, and via the parameter η it pins down the ratio of bequest to consumption during old age. As a result, it governs the relative strength of the two motives for saving during youth: life cycle considerations (c_{2t+1}) and bequest (b_{t+1}).

After some tedious calculations, it can be shown that the law of motion for the capital stock is given by

$$K_{t+1} = \frac{1+\eta}{2+\rho+\eta}[\frac{\eta}{1+\eta}(1+r_t)K_t + w_t L_t]. \tag{8}$$

Here total income of the young in period t equals their labor income $w_t L_t$ plus the bequest given from the old at time t. They receive a share $\eta/(1+\eta)$ of their parents total wealth which equals

[6]If households behave dynastically, i.e. derive utility from the utility of their offspring (cf Barro, 1974), the model allows for two general outcomes: either the household chooses to pass on bequest (if they attach sufficient weight on the welfare of future generations), or it does not. In the former case, the economy essentially works like an RCK model and the life cycle dimension effectively "washes out" from the perspective of capital accumulation. In the latter case, where bequest are not given, the economy is indistinguishable from a standard Diamond (1965) model in terms of capital accumulation. In the present case, we are interested in allowing simultaneously for the life cycle and bequest motive to be present, which is why we opt for a "joy-of-giving" setup. Dalgaard and Jensen (2009) adopt a similar utility function.

$(1+r_t)K_t$. The savings rate of the young is $(1+\eta)/(2+\rho+\eta)$. Naturally, the expression collapses to the standard Diamond equation (4) if $\eta = 0$. The consequence of introducing bequest is that capital income also matters to capital accumulation. When $\eta > 0$ not all savings from period $t-1$ are consumed in period t and consequently the size of the capital stock matters for future savings conditional on wage income.

Maintaining the assumption of Cobb-Douglas production, it can be shown that the model allows for a unique (and stable) steady state, where the capital-output ratio is

$$(\frac{y}{k})^* = \frac{(2+\rho+\eta)(1+g)(1+n)-\eta}{\eta\alpha+(1+\eta)(1-\alpha)} + \frac{\eta}{\eta\alpha+(1+\eta)(1-\alpha)}\delta.$$

The important new element is the second term which dictates that when depreciation increases, a larger share of total savings are used to maintain the steady-state capital stock, for which reason the capital-output ratio declines. Or, equivalently, the average product of capital increases. Hence, the countervailing effect from depreciation on the net marginal product of capital, discussed in the context of the Solow-Swan and RCK models, is now present.

Since the real rate of return is pinned down via the first order condition from firm maximization (equation (1)), we now have:

$$r^* = \alpha\frac{(2+\rho+\eta)(1+g)(1+n)-\eta}{\eta\alpha+(1+\eta)(1-\alpha)} - \frac{(1+\eta)(1-\alpha)}{\eta\alpha+(1+\eta)(1-\alpha)}\delta. \tag{9}$$

In the bequest-augmented model the influence from δ on the real rate of interest is no longer 1:1. Instead, it depends on the size of $(1+\eta)(1-\alpha)/(\eta\alpha+(1+\eta)(1-\alpha)) < 1$.

Why is the result different from the baseline Diamond model? When δ changes it affects the distribution of income between agents. In particular, it reduces the real income of the old. In the Diamond model their marginal propensity to consume from current income is one, whereas an operative bequest motive makes it strictly less than one in the present model, since a higher δ makes the old generation leave a smaller bequest for the young. As a consequence, savings are reduced and therefore the capital-output ratio declines. However, the resulting increase in the (gross) marginal productivity of capital is not enough to fully compensate for the direct negative effect of δ on the net marginal product.

In closing, a couple of final remarks on the implications of changing δ, as seen through the lens of the present model, are worthwhile. First, as also discussed in Section 3.3, the present model holds the implication that a higher rate of depreciation not only lowers the natural rate of interest but also produces a gap between the gross marginal product and r^*. Second, the bequest augmented Diamond model also suggests that rising δ should lead to a labor productivity slowdown. As higher δ lowers the capital-output ratio, it lowers steady-state labor productivity $(Y/L)^*$. In transition from one steady state to the next, labor productivity growth will therefore be lower than its trend level $(1+g)$, for standard convergence reasons. The length of time where the economy experiences sluggish growth depends on the rate of convergence. Empirical estimates suggest that the transition period may well be very lengthy (Barro, 2015). Hence, to the extent that the period during which the real rate has been declining also has also featured sluggish growth, this is not inconsistent with the predictions of the bequest-augmented Diamond model.

4.2 Gauging empirical magnitudes

In order to assess the impact of δ on r^*we need to pin down a reasonable value for η. In this respect it is useful to note that the model predicts that the fraction of the total capitalized value of the capital stock that is made up of bequest is constant at all points in time:

$$\frac{B_t}{(1+r_t)K_t} = \frac{\eta}{1+\eta}. \tag{10}$$

Recently, Alvaredo, Garbinti and Piketty (2017) estimate that $B_t / [(1+r_t)K_t]$ falls in the $0.5 - 0.6$ range for the US and Europe today. Hence, a value of η slightly above 1 is reasonable. If we take $\eta = 1$ and furthermore assume that the share of capital $\alpha = 0.4$, the "dampening factor" on the depreciation rate in equation (9) is

$$\frac{2(1-0.4)}{0.4+2(1-0.4)} = 0.75.$$

Hence, if δ increases by 1 percentage point, it would reduce the steady-state real rate of return by about 75 bps. Naturally, as discussed in Alvaredo et al. (2017) there is some uncertainty about the exact size of η.

In order to develop a more robust sense of magnitudes, it is useful to note that the model allows us to bound the influence from depreciation. When bequests are unimportant - $\eta = 0$ - we recover the 1:1 association from the Diamond setting. At the other extreme, when bequests become the key savings motive, we obtain a dampening factor on the influence from depreciation. Specifically

$$\lim_{\eta \to \infty} \frac{(1+\eta)(1-\alpha)}{\eta\alpha + (1+\eta)(1-\alpha)} = 1 - \alpha.$$

Hence, given $\alpha = 0.4$, the bequest-augmented Diamond model predicts that if δ rises by one percent, the steady-state real rate declines by 60 bps. Accordingly, the influence from depreciation is bounded in an interval from 0.6 to 1, with 0.75 as a reasonable best guess.

Where does this leave us in accounting for post-1970 developments in the real rate of return? The recent work by Rachel and Summers (2019) estimate for the group of Advanced Economies that the natural rate of interest has declined by about 300 bps since 1970. As discussed earlier, it is difficult to find comparable numbers for the Advanced Economies as a whole, but if developments in the US is a reasonable guide, the depreciation rate has increased by about one percentage points since 1970 (cf Section 2). This should, according to the model above, thus serve to reduce the real rate by between 60 bps and 100 bps or what amounts to between 1/4 to 1/3 of the secular decline in the natural rate.

5 External validity of the role played by depreciation

Recently evidence has started to accumulate that the post-1970 trend in the real rate might be part of a "super-secular" pattern. Schmelzing (2019) shows that carefully assembled data for the evolution of the real rate of interest exhibit a persistent negative trend since the 14th century. During the post-Napoleonic period (1820-), for example, Schmelzing documents that the trend amounts to an annual decline of 2.29 bps per year on average, which accumulates to a 4.6 pp decline in the real rate until present day.

As observed in the Introduction, various factors have been put forward as candidate explanations for the recent evolution of the real rate of interest, including changes in productivity growth, demographics, and more. Yet, the literature has quested the external validity of some of the explanations bearing the longer run perspective in mind. For instance, Borio et al. (2017), using panel data from 1870 onward for 19 countries, find that several variables believed to be key drives of changes in the real rate often seem not to co-vary with real rates in the expected way. A similar pattern is uncovered by Lunsford and West (2019) who explore the determination of the real rate in the United States since 1890 using time series methods. Though some standard demographic determinants are correlated with the real rate in the expected way most other variables exhibit a mixed relationship to the real rate.

This raises the question of whether the mechanism under scrutiny suffers an "external validity" issue? To put it differently: Is there a reason to believe that depreciation could have played a role in the long-run pattern uncovered by Schmelzing (2019), in addition to the more recent (post-1990) decline in the real rate?

Answering the latter question rigorously would require data on the composition of the capital stock and estimates of the depreciation rate by asset class for the early 19th century. Perhaps future work by economic historians will produce this kind of data. For now, we will have to resort to more circumstantial evidence.

The point we will make is that since the relative strength of the two savings motives in focus here (life cycle vs. bequest) likely changed over the period the *pass through* of the depreciation rate likely changed as well. This change should put a downward force from the depreciation rate on the real rate *even if* the depreciation rate itself remained constant over the period (see equation 9). We reiterate that our point is that the strength of the negative force *from depreciation* on the real rate increases when the savings motive changes. Whether the structural shift in the savings motive *overall* has served to lower the real rate is a different proposition that we do not posit; we only focus on the partial influence from the depreciation rate.

DeLong (2003) argues that before the Industrial Revolution bequest played a much more crucial role in the process of capital accumulation than it does today. More concretely, Delong argues that while the share of bequest in total wealth is perhaps about 0.4 today (a bit lower than the more recent estimate by Alvaredo et al. (2017)) the figure is likely to have been closer to 0.9 in the pre-industrial world. Broadly consistent with these views Alvaredo et al. (2017) find a bequest share of about 0.5 in 21st century Europe and between 0.7 and 0.8 in 1900. In the US, the pattern is more ambiguous and seems to exhibit a wave-like pattern from 1900 to 2010, starting and ending at a bequest share between 0.5 and 0.6.

Suppose we take DeLong's estimate for the pre-industrial setting as a rough proxy for the situation in the early 19th century, and the recent estimates from Alvaredo et al. (2017) as our end point estimate for the share of bequest. Considering equation (9) we would then expect the "depreciation channel" to be modified in the following way:

$$\Delta r^* = -\left[\frac{(1+\eta^{2020})(1-\alpha)}{\eta^{2020}\alpha + (1+\eta^{2020})(1-\alpha)} - \frac{(1+\eta^{1820})(1-\alpha)}{\eta^{1820}\alpha + (1+\eta^{1820})(1-\alpha)}\right]\delta,$$

assuming a constant rate of depreciation throughout the period and share of capital. Now, suppose the share of capital is 0.4, and note that Delong's calculations suggest a value for $\eta^{1820}$of about 9, while Alvaredo et al. (2017) suggest a value of $\eta^{2020}$of about 1. This means a change in the partial effect of the depreciation rate of $\Delta r^* \approx -0.125\delta$. To obtain an implied decline in the real rate we need a reasonable guess for average depreciation over the period, which we cannot know. But if we disregard the recent run-up in δ, documented above, a value of about 3.6 percent may be reasonable. In this case, theory would suggest a resulting decline in the real rate of roughly 45 basis points, which is about ten percent of the observed decline since 1820, according to the data assembled by Schmelzing (2019).

Naturally, if the average depreciation rate in the 21st century is *higher* than during the 19th century, this number will increase. Across this time horizon, it may well have increased due to changes in the composition of assets. Going sufficiently far back in time, most of the capital stock would probably be structures. Over time equipment would matter progressively more, just as ICT-related capital assets have risen in importance recently. Assessing the quantitative significance of such changes for the real rate - if they occurred - will, however, have to await the availability of the relevant data.

From this discussion two conclusions emerge. First, the mechanism that we have explored in the present paper is consistent with long-term trends. Second, while the mechanism may contribute to an understanding of the forces that have been responsible for a "suprasecular" decline in the real rate of interest, it leaves much unexplained.

6 Concluding remarks

The present study draws attention to a hitherto overlooked factor which may have contributed to a declining real rate of return on safe assets over the long run: a rising rate of capital depreciation. Judged from the best available evidence, the rate of capital depreciation has increased over the last half century in the US, and it seems likely that this trend is pervasive across advanced economies. A decomposition analysis finds that both a rising share of capital assets featuring relatively high capital depreciation, such as ICT and software, and a rising depreciation rate *within* asset types is responsible. If capital accumulation is sensitive to capital income to a degree that is empirically realistic a rising rate of capital depreciation can account for a non-negligible reduction in the real rate of interest.

The present study raises several questions worth exploring in future research. The first issue concerns the apparent rise in the rate of capital depreciation itself. The Bureau of Economic Analysis produces data where the rate of depreciation is allowed to vary within asset classes. But this approach is not followed in most countries around the world. In general the assumption appears to be that of a fixed depreciation rate within asset categories making it difficult to assess how the average depreciation rate has moved intertemporally. Consequently, more data work is needed to gauge how pervasive the movements in the rate of depreciation are across countries.

If the trend is as pervasive as it appears to be, the second issue is why the rate of depreciation has risen. To the extent that the trend is caused by a changing composition of assets, especially towards ICT and software, it would seem to be an interesting by-product of the "IT revolution". But if the trend is (also) due to rising depreciation *within* assets categories the explanation is less straight forward. In principle many possible explanations can probably be developed. But one theoretically plausible candidate explanation would be that a declining relative price of new capital goods (a salient fact in its own right) might have induced firms to reduce the extent of maintenance of existing capital equipment leading to faster depreciation (e.g., McGrattan and Schmitz, 1999). If so the role played by a declining relative price of investment in accounting for the secular decline in the real rate, which currently is considered to be non-negligible but modest (Sajedi and Thwaites, 2016), should be re-examined.

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Appendix

A. Weighing different types of capital

In the following we demonstrate that when there are several different types of capital present — with distinct economic and physical depreciation rates — the relevant aggregator is one that aggregates economic and physical capital using the nominal stock of capital at each point in time.

OLG model with economic depreciation and two types of capital

Let's consider two types of capital, one with economic depreciation and one without. It turns out that nominal capital is in fact the right way to do the weighted average of depreciation rates.

Setup: Overlapping generations.

We continue with the OLG setup, normalize the stock of labor to $L = 1$ and let their be two types of capital.

$$Y_t = K_{1,t}^{\alpha_1} K_{2,t}^{\alpha_2},$$

with $\alpha_1, \alpha_2 > 0$.

The production function implies marginal product of capital of:

$$MPK_{1,t} = \alpha_1 K_{1,t}^{\alpha_1 - 1} K_{2,t}^{\alpha_2},$$

$$MPK_{2,t} = \alpha_2 K_{1,t}^{\alpha_1} K_{2,t}^{\alpha_2 - 1}.$$

There is a production technology which makes final output into capital type 2 one-for-one. We normalize the price of output $p^Y = 1$ and will consequently get $p^{K_2} = 1$. The physical depreciation of type 2 is δ_2 and the capital accumulation follows:

$$K_{2,t+1} = I_{2,t} + (1 - \delta_2) K_{2,t},$$

where $I_{2,t}$ is units of final good invested in capital type 2 good. There is technological improvement in the quality of type 1 capital. Specially, one unit of output creates $(1 + \lambda)^t$ units of capital of type 2 in period t.

The capital accumulation function for capital of type 2 is:

$$K_{1,t+1} = (1+\lambda)^t I_{1,t} + (1-\delta_1)K_{1,t},$$

where $I_{1,t}$ is the units of final good spent on investment in type 1.

We will conjecture that there is a steady state where output, and capital stocks grow at g^Y, g^{K_1} and g^{K_2}, respectively.

This implies:

$$I_{1,t} = (1+g^Y)^t I_{1,0},$$
$$I_{2,t} = (1+g^Y)^t I_{2,0},$$
$$Y_t = (1+g^Y)^t Y_0,$$

where $I_{1,0}$,$I_{2,0}$ and Y_0 are some constants. Obviously, I will need to make sure that this can be an equilibrium below.

This implies:

$$K_{1,t+1} = \left[(1+\lambda_1)(1+g^Y)\right]^t K_{1,1}. \tag{11}$$

$$K_{2,t+1} = (1+g^Y)^t K_{2,1}. \tag{12}$$

And with the growth rate in output of:

$$1+g^Y = \frac{Y_{t+1}}{Y_t} = \left(\frac{K_{1,t+1}}{K_{1,t}}\right)^{\alpha_1}\left(\frac{K_{2,t+1}}{K_{2,t}}\right)^{\alpha_2}$$

$$1+g^Y = (1+\lambda_1)^{\frac{\alpha_1}{1-\alpha_1-\alpha_2}}.$$

We keep preferences logarithmic and continue to have that young people save:

$$s_t w_t = \frac{1}{2+\rho} w_t.$$

The equilibrium requires a set of arbitrage conditions: First, let the interest rate be given by r_t. Then one dollar invested in asset type 2 must give the same return as bonds. Since capital of type 2 retains its value the arbitrage condition requires:

$$1+r_t = MPK_{2,t} + 1 - \delta_2.$$

A dollar invested in capital of type 1 can buy $q_t = (1+\lambda)^t$ units at price $p_t = (1+\lambda)^{-t}$. That will give $MPK_{1,t} \times q_t$ in return. The price of the physical asset will depreciate by $p_{t+1}q_{t+1}/(p_t q_t) = \frac{1-\delta}{(1+\lambda)}$ such that we require:

$$1+r_t = (1+\lambda)^t MPK_{1,t} + \frac{1-\delta}{1+\lambda}.$$

Employing the growth rates of capital stock from above one can show that both $MPK_{2,t}$ and $(1+$

$\lambda)^t MPK_{1,t}$ remain constant.

Consequently, arbitrage requires two conditions:

$$1 + r_t = MPK_{2,t} + 1 - \delta_2, \tag{13}$$

$$1 + r_t = (1+\lambda)^t MPK_{1,t} + \frac{1-\delta_1}{1+\lambda}. \tag{14}$$

Weighing by nominal capital

We first weigh equations (13) and (14) the two expressions by nominal capital: $(1+\lambda)^{-t}K_{1,t}$ and $K_{2,t}$:

$$1 + r_t = \frac{K_{2,t}MPK_{2,t} + (1+\lambda)^{-t}K_{1,t}MPK_{1,t}}{K_{2,t} + (1+\lambda)^{-t}K_{1,t}} + \frac{K_{2,t}(1-\delta_2) + (1+\lambda)^{-t}K_{1,t}\frac{1-\delta_1}{1+\lambda}}{K_{2,t} + (1+\lambda)^{-t}K_{2,t}}.$$

Such that

$$(1+r_t) = \frac{\alpha_1 K_{1,t}^{\alpha_1}K_{2,t}^{\alpha_2} + \alpha_2 K_{1,t}^{\alpha_1}K_{2,t}^{\alpha_2}}{(1+\lambda)^{-t}K_{1,t} + K_{2,t}}$$

$$+\frac{\frac{1-\delta_1}{1+\lambda}(1+\lambda)^{-t}K_{1,t} + (1-\delta_2)K_{2,t}}{(1+\lambda)^{-t}K_{1,t} + K_{2,t}}.$$

Note that:

$$\alpha_1 K_{1,t}^{\alpha_1}K_{2,t}^{\alpha_2} + \alpha_2 K_{1,t}^{\alpha_1}K_{2,t}^{\alpha_2} = Y - wL,$$

i.e. total return to capital. Let $K_t \equiv (1+\lambda)^{-t}K_{1,t} + K_{2,t}$ be the nominal value of capital. So I am getting that:

$$(1+r_t) = MPK \times K_t + (1-\hat{\delta}),$$

where MPK is the (constant) average marginal product of capital and $\hat{\delta}$ is the weighted sum of economic and physical depreciation where the weight is the nominal stock of capital. Consequently, weighing by nominal stock reproduces a model with a single stock of capital.

Note, that $(1+\lambda)^{-t}K_{1,t}$ where:

$$K_{1,t} =$$

$$K_{1,t+1} = (1+\lambda)^t I_{1,t} + (1-\delta_1)K_{1,t} = (1+\lambda)^t I_{1,t} + (1-\delta_1)\left[(1+\lambda)^{t-1}I_{1,t-1} + (1-\delta_1)K_{1,t-1}\right]$$

$$\sum_{s=-\infty}^{t}(1+\lambda)^s(1-\delta_1)^{t-s}I_{1,s} = (1+\lambda)^t \sum_{s=-\infty}^{t}(1-\delta_1)^{t-s}\frac{I_{1,s}}{(1+\lambda)^{t-s}},$$

that is capital measured by the perpetual inventory method where investment in period s is deflated by the price index to year t .

Weighing by real stock of capital

We briefly outline why weighing by real stock of capital is incorrect. Above we outlined a steady state in which the relevant element of the savings decision, and in particular the depreciation rate,

$\hat{\delta}$, remain constant. Instead, let $\tilde{\delta}_{t,s}$ be the depreciation rate of year t measured from the perspective of year s.

We can then write:

$$1 - \tilde{\delta}_{t,s} = \frac{K_{2,t,s}(1-\delta_2) + K_{1,t,s}\frac{1-\delta_1}{1+\lambda}}{K_{2,t,s} + K_{1,t,s}},$$

which we combine equations (11) and (12) to get:

$$K_{1,t,s} = K_{1,s,s}\left[(1+\lambda_1)(1+g^Y)\right]^{t-s},$$

$$K_{2,t,s} = K_{2,s,s}(1+g^Y)^{t-s},$$

which gives:

$$1 - \tilde{\delta}_{t,s} = \frac{K_{2,s,s}(1-\delta_2) + K_{1,s,s}\left[(1+\lambda_1)\right]^{t-s}\frac{1-\delta_1}{1+\lambda}}{K_{2,s,s} + \left[(1+\lambda_1)\right]^{t-s}K_{1,s,s}},$$

where $\delta_{t,s}$ is increasing in t if $\frac{1-\delta_1}{1+\lambda} < 1 - \delta_2$. Comparing equipment with structures this condition holds and mechanically the depreciation rate will be lower in the past ($t < s$) even in a steady state equilibrium.

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B. The real rate in the Diamond model with more general preferences

Suppose per period preferences are slightly more general CRRA preferences:

$$u = \frac{c_{1t}^{1-\theta} - 1}{1-\theta} + \frac{1}{1+\rho}\frac{c_{2t+1}^{1-\theta} - 1}{1-\theta}$$

The savings rate is now a function of the (future) real rate of interest, $s\left(r_{t+1}\right)$. For existence and uniqueness of a self-fulfilling expectation path we require (cf Galor and Ryder, 1989, lemma 1):

$$s_r \geq 0.$$

The law of motion of the capital stock in efficiency units (since we maintain Cobb-douglas) is

$$k_{t+1} = \frac{1}{\left(1+n\right)\left(1+g\right)}s\left(r_{t+1}\right)\left(1-\alpha\right)k_t^{\alpha}$$

, which means that we can write the steady state

$$\left(\frac{k}{y}\right)^* = \frac{\left(1-\alpha\right)}{\left(1+n\right)\left(1+g\right)}s\left(r^*\right),$$

since the law of motion is fulfilled at all points in time, the steady state included.

This gives combinations of the capital-output ratio and the real rate of interest that are consistent with a (unique) steady state. The schedule is monotonically upward sloping due to the requirement imposed by existence and uniqueness self-fulfilling expectation path,$s_r \geq 0$ for all r. The intuition is that a higher real rate of interest spurs savings and fosters a higher capital-output ratio in the steady state.

The other key equation is the FOC from the firm

$$r = \alpha \left(\frac{k}{y} \right)^{-1} - \delta$$

which gives combinations of capital-output ratios and the real rate that are consistent with clearing in the market for capital. Its downward sloping implying a unique intersection point between the two schedules.

Plotting the two schedules i an (K/Y, r) diagram, and perturbing the depreciation rate, immediately leads to the insight that in the steady state the capital-output ratio is lowered, but so is the real rate of interest. Hence, the downward adjustment in the capital stock does not fully compensate for the direct impact of a rising depreciation rate. The case discussed in the text is the special case where $s_r = 0$. In this case the decline in r is larger since there is no adjustment to the capital-output ratio when the depreciation rate changes – the steady state schedule is vertical, featuring a fixed capital-output ratio.